STiCM

Select / Special Topics in Classical Mechanics

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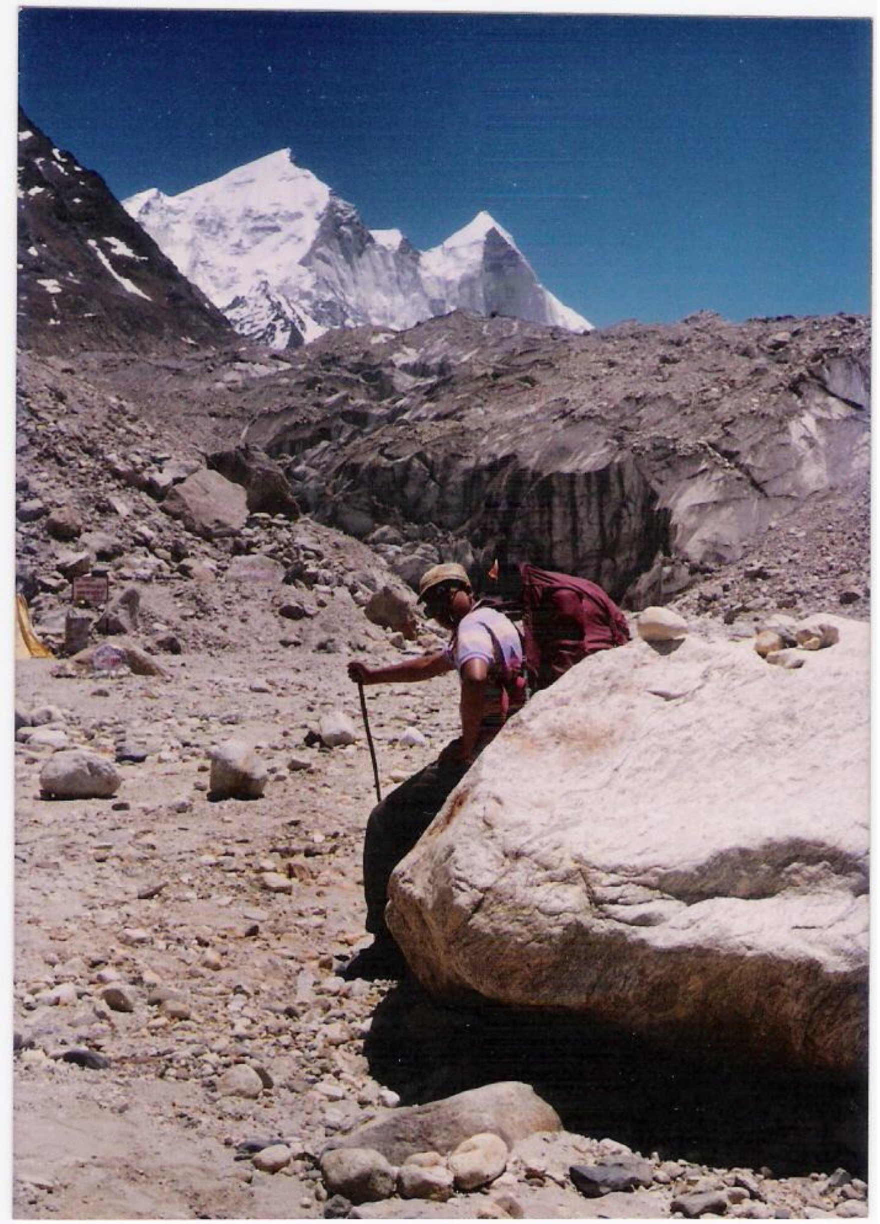
STiCM Lecture 23

Unit 7 : Potentials, Gradients, Fields

Unit 7: Physical examples of fields. Potential energy function. Gradient, Directional Derivative,



Learning goals:

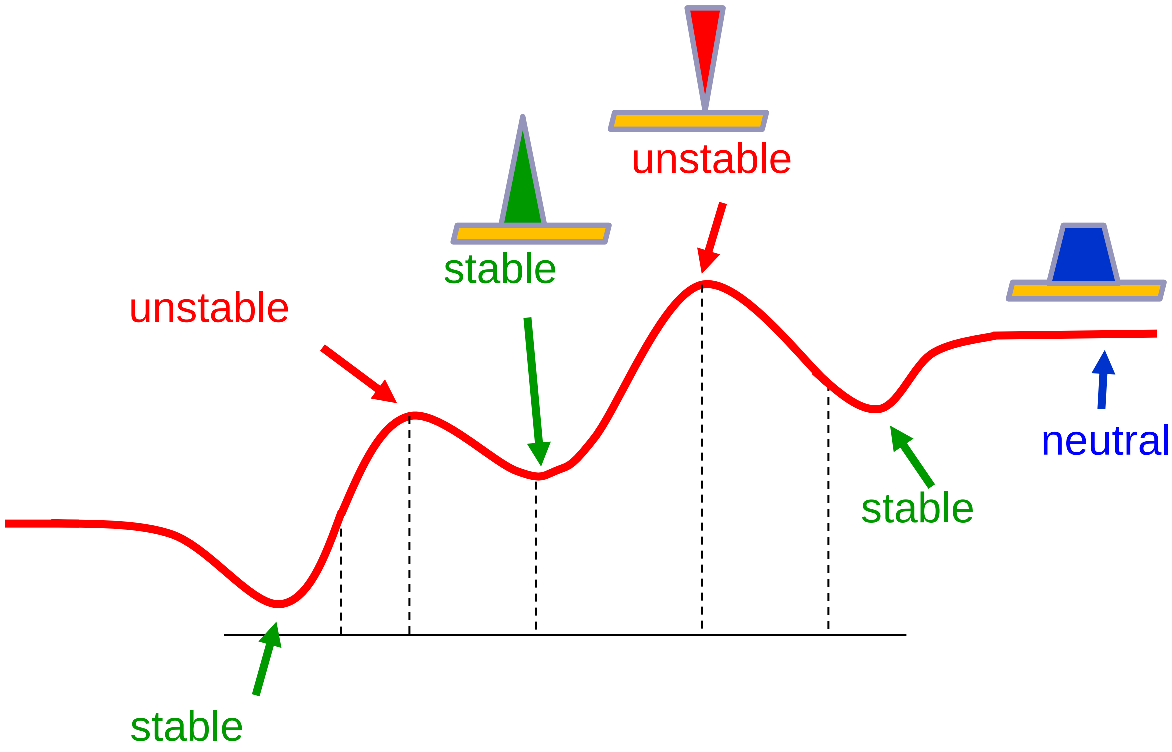
Develop a strong handle on methods of vector calculus that provide powerful tools to study the relationships between physical potentials and field.

GOMUKH

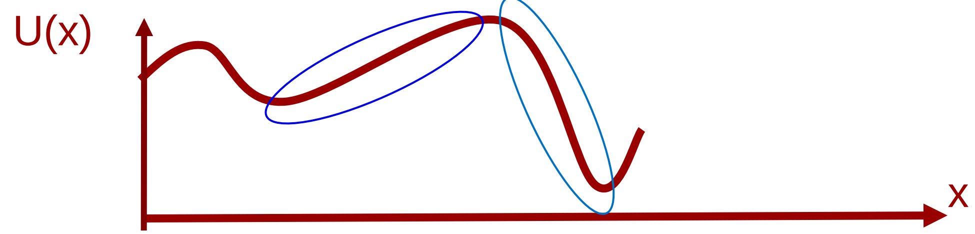
BHAGIRATH peak

BHAGIRATHI / GANGA

Kinds of equilibrium



$$F = -\frac{dU}{dx} = ma$$



The slope determines the acceleration that would result.

How is 'potential' related to 'field' in 3-dimensions ?

$$\vec{F} = -\vec{\nabla} U$$

$\vec{\nabla}$: gradient operator / nabla / del

What is meant by ‘potential’ ?

- *Some kind of capability*

Newtonian concepts:

1) Equilibrium : self sustaining – I law

2) Departure from equilibrium

– requires net force/interaction/cause

- results in acceleration

If an object moves to a region where its potential changes, its equilibrium is disturbed.

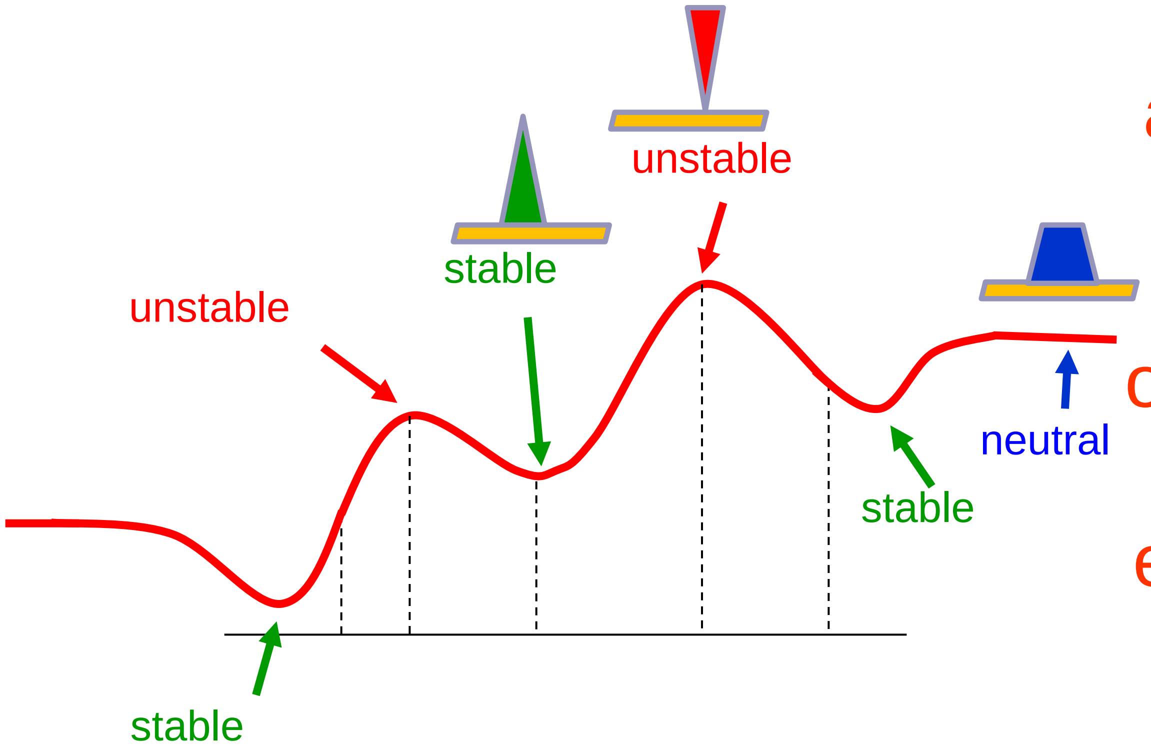
What is meant by ‘field’ ?

Agency that disturbs equilibrium.

Cognizable effect: acceleration, change in equilibrium.

Field: 'agency' that produces the acceleration.

When 'potential' changes, equilibrium is disturbed.



No acceleration takes place when an object moves along an equipotential path.

Common examples of potential, field

Gravitational

Electromagnetic

Elastic

Chemical

Torsion

.....

Alternative formulation of 'MECHANICS'

The mechanical system evolves in such a way that

'*action*', $S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt$ is an extremum

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \quad \textit{Lagrange's Equation} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$L = L(q, \dot{q}, t) = L(q, \dot{q})$

when the Lagrangian is independent of 'time'

Homogeneity & Isotropy of space

Lagrangian L can only be quadratic function of the velocity.

$$L(q,\dot{q}) = f_1(\dot{q}^2) + f_2(q)$$

$$L(q,\dot{q},t) = \frac{m}{2}\dot{q}^2 - V(q)$$
$$= T - V$$

$$L(q,\dot{q},t) = \frac{m}{2}\dot{q}^2 - V(q) = T - V$$

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} = F, \text{ the force}$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} = p, \text{ the momentum}$$

Interpretation of L as T-V gives equivalent correspondence with Newtonian formulation.

In 3-dimensional configuration space:

$$\vec{F} = -\vec{\nabla} U$$

Physical Universe is made up of Particles and Fields and their mutual interactions

- Particles: Material world

- Fields: 'Action at a distance' - involved concept

- implications in classical mechanics / gravity, EM field

- implications in quantum physics (Non-Locality; EPR paradox)

Quantitative / Mathematical relation between POTENTIAL and FIELD

Potentials → Fields

Differentiation

←

Integration / Constant of Integration

Boundary Value problem

"It is, as Schrodinger has remarked, a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered.

……...

….. The laws of nature are concerned with such regularities."

Eugene P.. Wigner: Communications in Pure and Applied Mathematics, Vol. 13, No. I (February 1960). THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

We should be grateful for it ……….

Eugene P.. Wigner: Communications in Pure and Applied Mathematics, Vol. 13, No. I (February 1960). THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

Mathematical Connections…..

Differentiation

Potentials

Fields

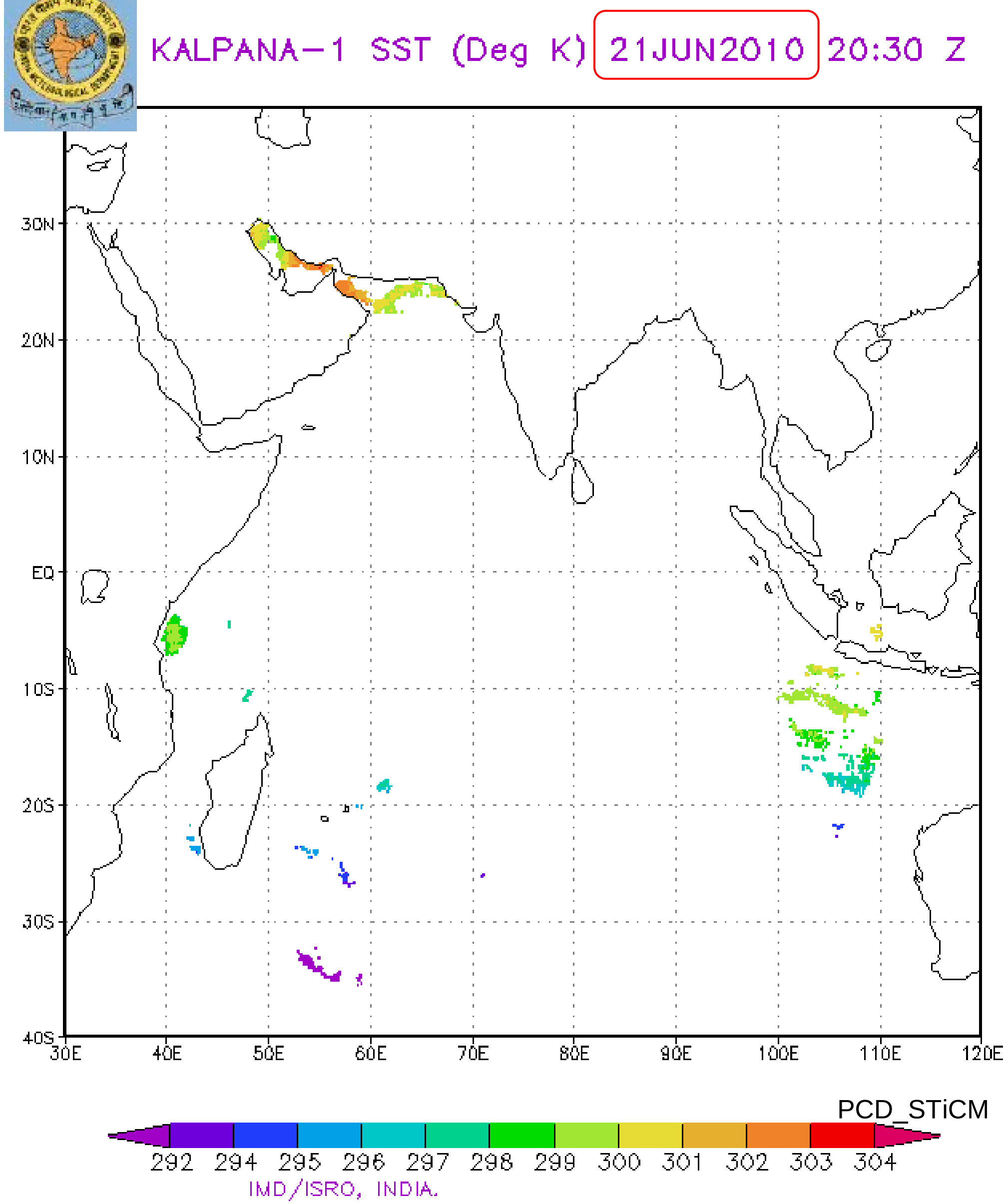
Integration / Constant of Integration

Boundary Value problem

SCALAR
POINT FUNCTIONS

VECTOR
POINT FUNCTIONS

Scalar 'Point' function



SST:

Sea Surface Temperature

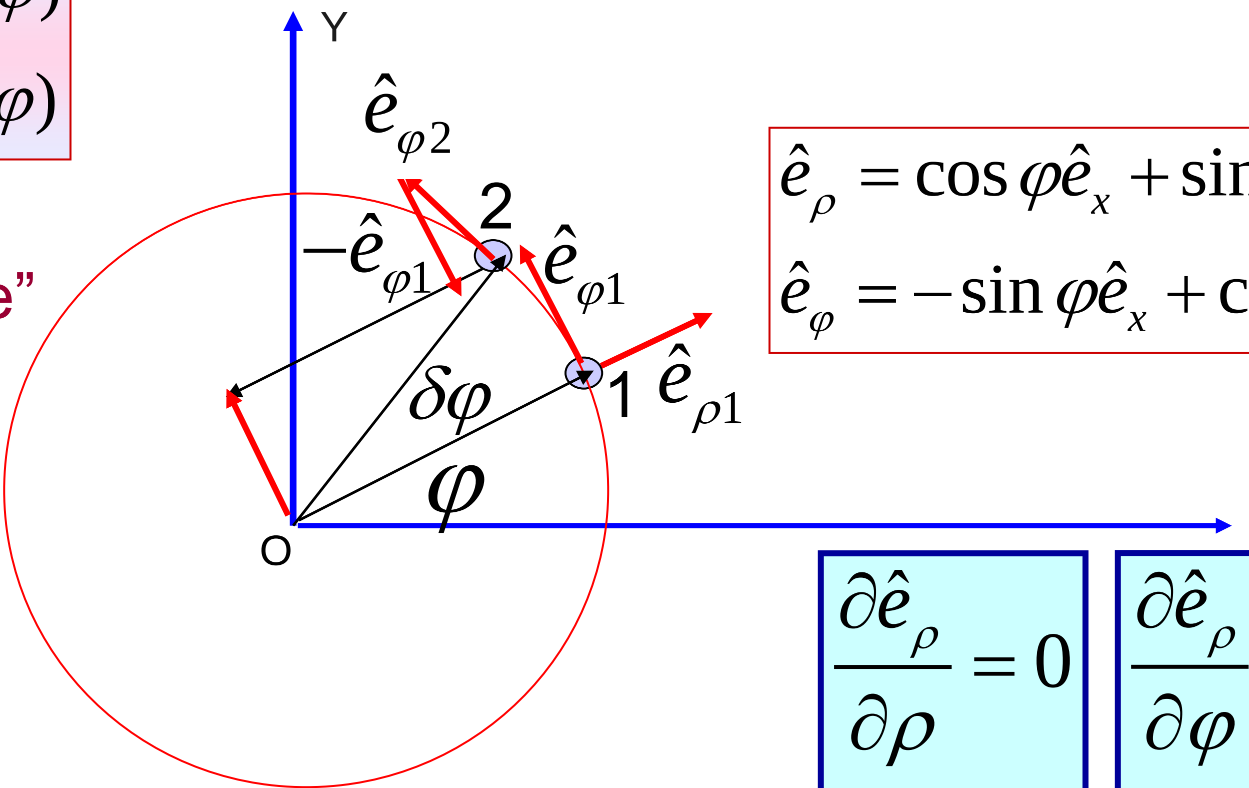
We see a 'property' that changes from point to point – it is a 'point function'

$(\hat{e}_\rho, \hat{e}_\varphi)$ are not constant vectors.

$$\psi = \psi(\vec{r}) = \psi(x, y, z)$$
$$= \psi(r, \theta, \varphi) = \psi(\rho, \varphi, z)$$

$$\hat{e}_\rho = \hat{e}_\rho(\not{\rho}, \varphi)$$
$$\hat{e}_\varphi = \hat{e}_\varphi(\not{\rho}, \varphi)$$

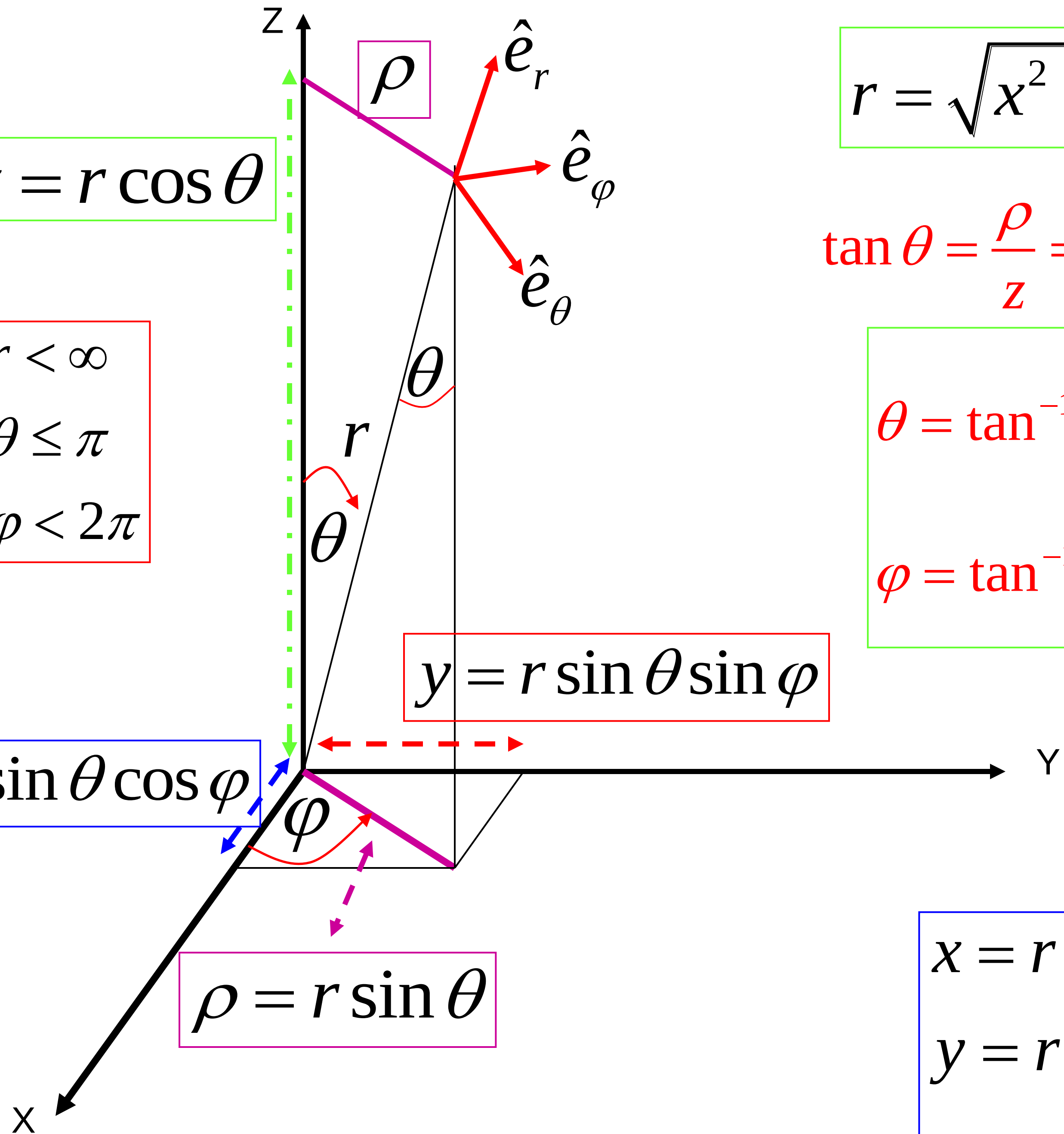
"Unit Circle"



$$\hat{e}_\rho = \cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y$$
$$\hat{e}_\varphi = -\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y$$

$$\frac{\partial \hat{e}_\rho}{\partial \rho} = 0 \qquad \frac{\partial \hat{e}_\rho}{\partial \varphi} = \hat{e}_\varphi,$$
$$\frac{\partial \hat{e}_\varphi}{\partial \rho} = 0 \qquad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$

$$\lim_{\delta\varphi \to 0} \frac{\hat{e}_{\varphi 2} - \hat{e}_{\varphi 1}}{\delta\varphi} = \lim_{\delta\varphi \to 0} \frac{\delta \hat{e}_\varphi}{\delta\varphi} = \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$



$$z = r\cos\theta$$

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \varphi < 2\pi$$

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$\rho = r\sin\theta$$

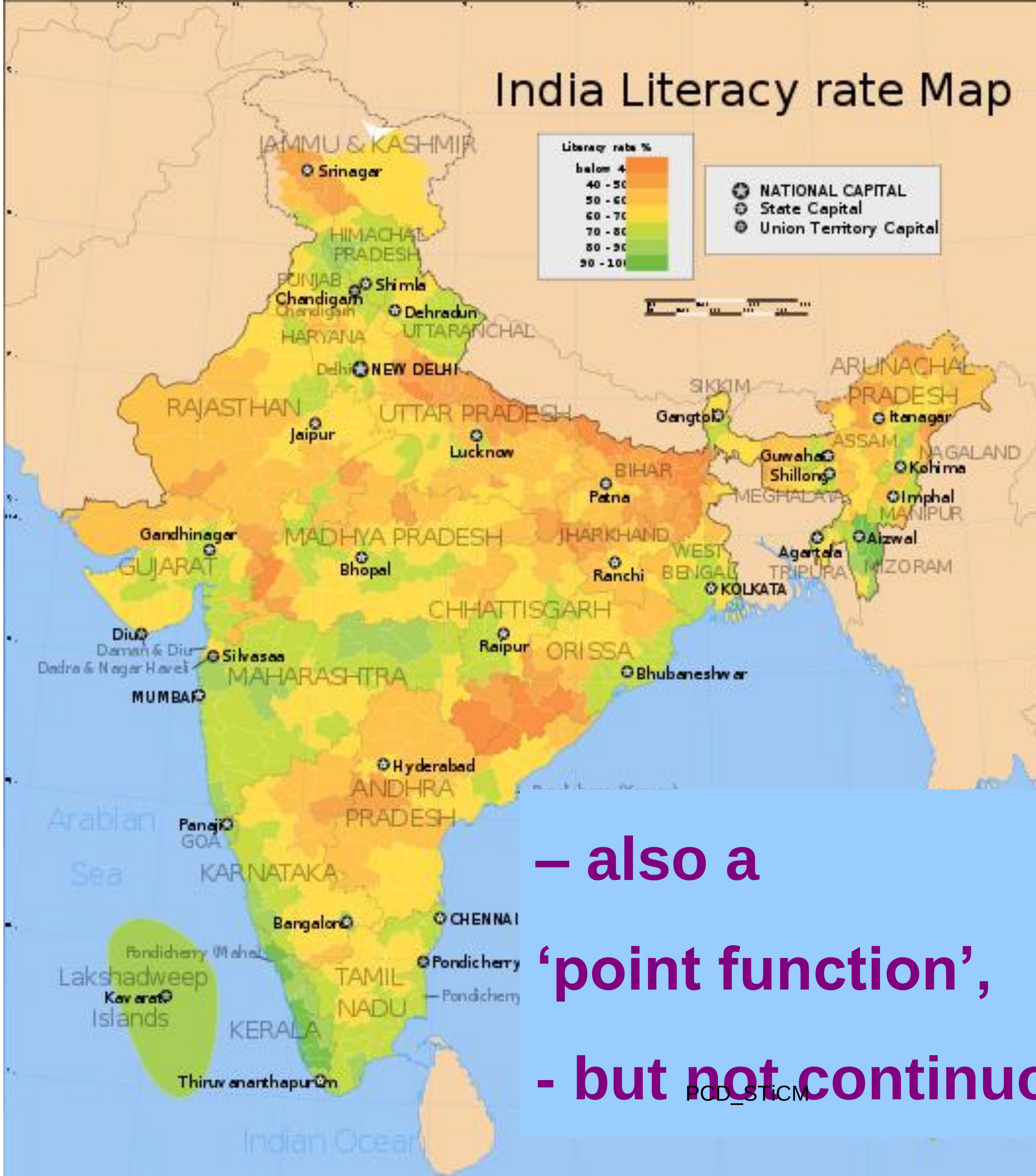
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\rho}{z} = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

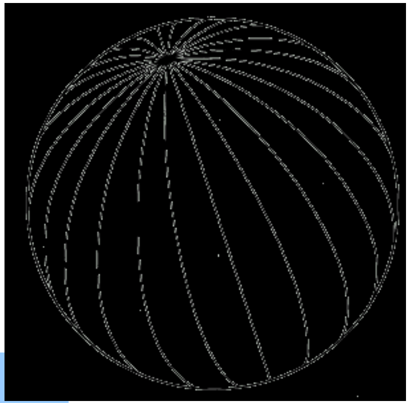
$$\varphi = \tan^{-1}\frac{y}{x}$$

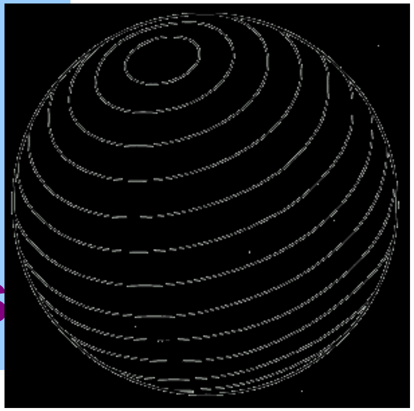
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$



Literacy %

L=L(x,y)





– also a

'point function',

- but not continuous

Vector Fields:
'Point'
function

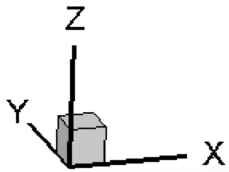
$$\vec{V} = \vec{V}(\vec{r}) = \vec{V}(x, y, z)$$

$$= \vec{V}(r, \theta, \varphi) = \vec{V}(\rho, \varphi, z)$$

$$\vec{V} = \vec{V}(\vec{r}, t)$$

discrete
versus
continuous

Vector Fields:
'Point'
function



$$\vec{V} = \vec{V}(\vec{r}) = \vec{V}(x, y, z)$$
$$= \vec{V}(r, \theta, \varphi) = \vec{V}(\rho, \varphi, z)$$

$$\vec{V} = \vec{V}(\vec{r}, t)$$

discrete
versus
continuous

Fluids: Continuum Model

Scalar/ Vector Fields: 'Point' function

$$\psi = \psi(\vec{r}) = \psi(x, y, z) = \psi(r, \theta, \varphi) = \psi(\rho, \varphi, z)$$

$$\vec{A} = \vec{A}(\vec{r}) = \vec{A}(x, y, z) = \vec{A}(r, \theta, \varphi) = \vec{A}(\rho, \varphi, z)$$

<u>Examples of Scalar Point Functions</u>

- temperature
- gravitational/electrostatic potentials
- pressure in a liquid column

<u>Examples of Vector Point Functions</u>

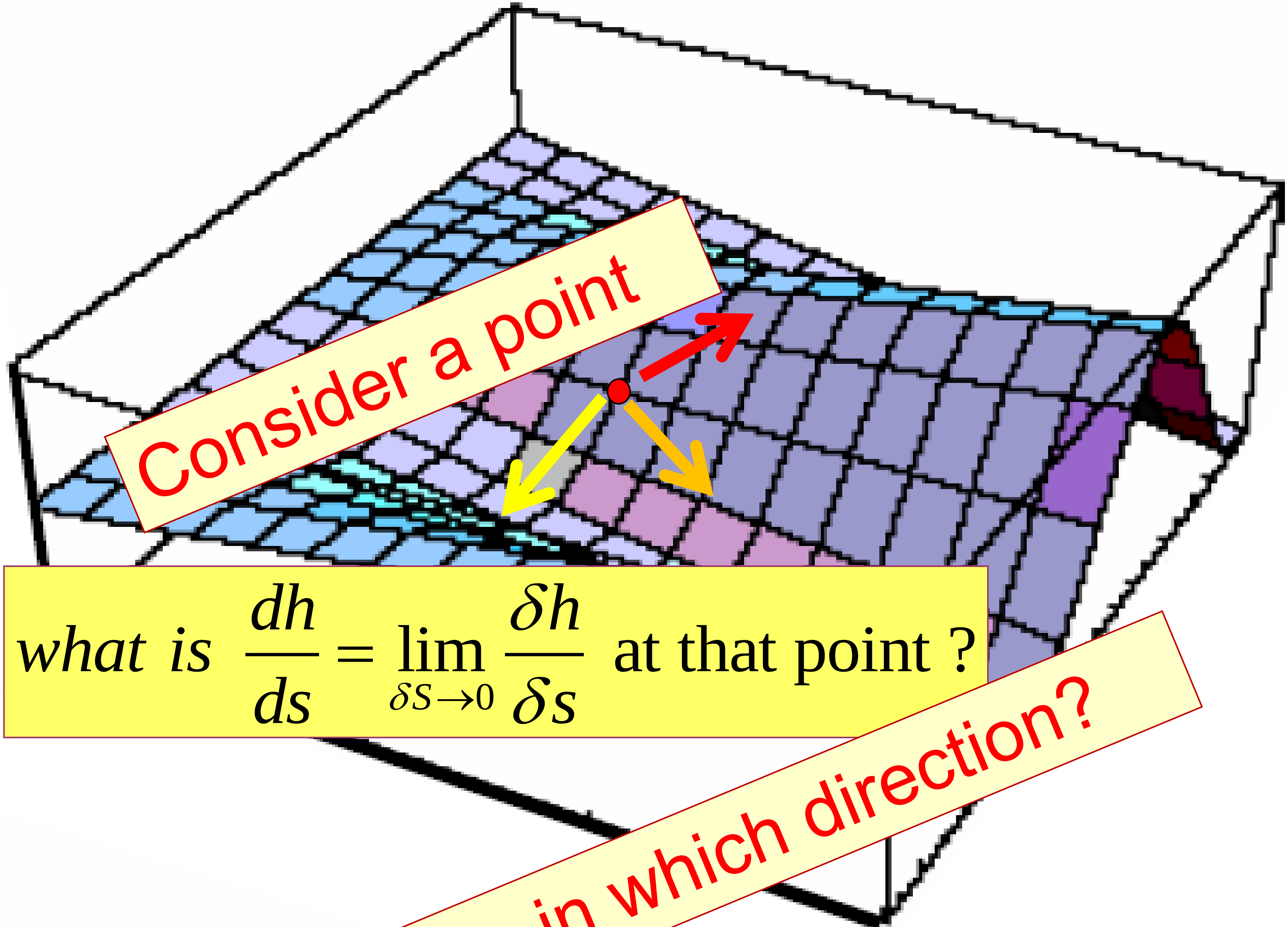
- velocity field
- electric field
- magnetic field

Scalar/ Vector Fields: 'Point' function

Functions of 'space' and 'time'

$$\psi = \psi(\vec{r}, t) = \psi(x, y, z, t) = \psi(r, \theta, \varphi, t) = \psi(\rho, \varphi, z, t)$$

$$\overrightarrow{A} = \overrightarrow{A}(\vec{r}, t) = \overrightarrow{A}(x, y, z, t) = \overrightarrow{A}(r, \theta, \varphi, t) = \overrightarrow{A}(\rho, \varphi, z, t)$$



Consider a point

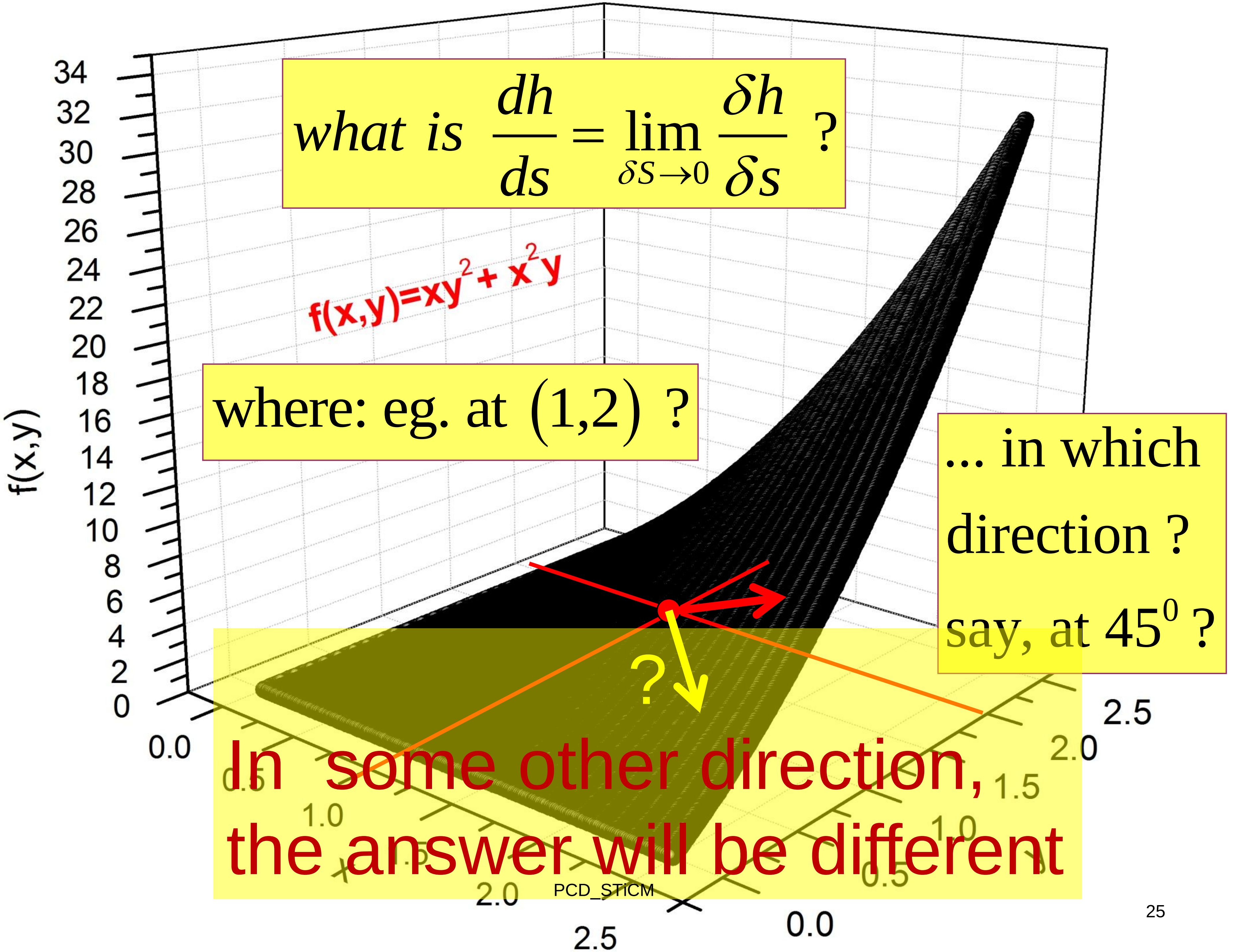
$$what\ is\ \frac{dh}{ds} = \lim_{\delta S \to 0} \frac{\delta h}{\delta s}\ \text{at that point ?}$$

....in which direction?

$$\text{what is } \frac{dh}{ds} = \lim_{\delta S \to 0} \frac{\delta h}{\delta s} \text{ ?}$$

where: eg. at $(1,2)$?

... in which direction ? say, at 45^0 ?



In some other direction, the answer will be different

DIRECTIONAL DERIVATIVE

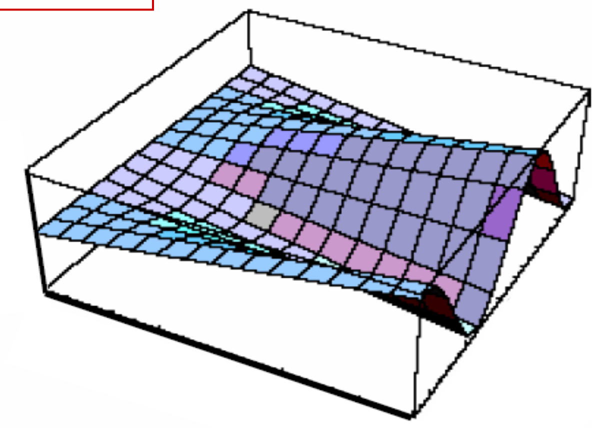
is a SCALAR QUANTITY

which has a DIRECTIONAL ATTRIBUTE.

The rate of change of ψ with distance s is a scalar given by

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi(\vec{r})}{\delta s} = \lim_{\delta s \to 0} \frac{\psi(\vec{r}+\delta\vec{r})-\psi(\vec{r})}{\delta s}$$

This 'rate' ('slope') depends on the direction in which the displacement $\delta\vec{r}$ is considered.



- **Ratio of two scalar quantities.**
- **It has a 'directional attribute'**

δs : displacement in which direction ? $\delta s = \left|\overrightarrow{\delta r}\right|$

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\psi(\vec{r} + \delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

$$\psi(\vec{\mathrm{r}}) = \psi(x, y, z)$$

$$\psi(\vec{\mathrm{r}}) = \psi(\rho, \varphi, z)$$

$$\psi(\vec{\mathrm{r}}) = \psi(r, \theta, \varphi)$$

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial z}\delta z$$

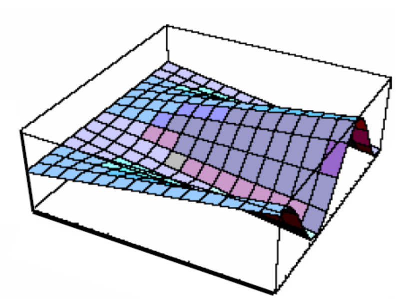
Cartesian Coordinate System

$$= \frac{\partial\psi}{\partial\rho}\delta\rho + \frac{\partial\psi}{\partial\varphi}\delta\varphi + \frac{\partial\psi}{\partial z}\delta z$$

Cylindrical Polar Coordinate System

$$= \frac{\partial\psi}{\partial r}\delta r + \frac{\partial\psi}{\partial\theta}\delta\theta + \frac{\partial\psi}{\partial\varphi}\delta\varphi$$

Spherical Polar Coordinate System



Expressions for $\delta\psi$

What about the expressions for $\frac{d\psi}{ds}$?

The directional derivative

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(\vec{r} + \delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

$$\psi(\vec{\mathrm{r}}) = \psi(x, y, z)$$

$$\psi(\vec{\mathrm{r}}) = \psi(\rho, \varphi, z)$$

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$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial z}\frac{dz}{ds}$$

← Cartesian Coordinate System

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial\rho}\frac{d\rho}{ds} + \frac{\partial\psi}{\partial\varphi}\frac{d\varphi}{ds} + \frac{\partial\psi}{\partial z}\frac{dz}{ds}$$

← Cylindrical Polar Coordinate System

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial r}\frac{dr}{ds} + \frac{\partial\psi}{\partial\theta}\frac{d\theta}{ds} + \frac{\partial\psi}{\partial\varphi}\frac{d\varphi}{ds}$$

← Spherical Polar Coordinate System

The directional derivative

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(\vec{r} + \delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

We develop

expressions in

Cartesian Coordinate System

Cylindrical Polar

Spherical Polar

The expressions turn out to be nice and simple, obtained easily by using the expressions for the displacement $\overrightarrow{dr}$ in various coordinate systems.

The directional derivative

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(\vec{r} + \delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

Expressions in various coordinate systems

Questions ? Comments ?

We shall take a break here…….

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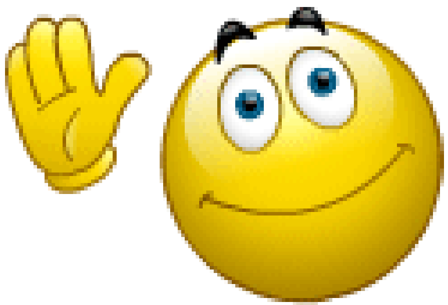
Next L24 : Unit 7 ………

The directional derivative: $\frac{d\psi}{ds}$

Potentials, Gradients, Fields…….



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STiCM Lecture 24

Unit 7 :

The directional derivative: $\frac{d\psi}{ds}$

Potentials, Gradients, Fields

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is a SCALAR QUANTITY

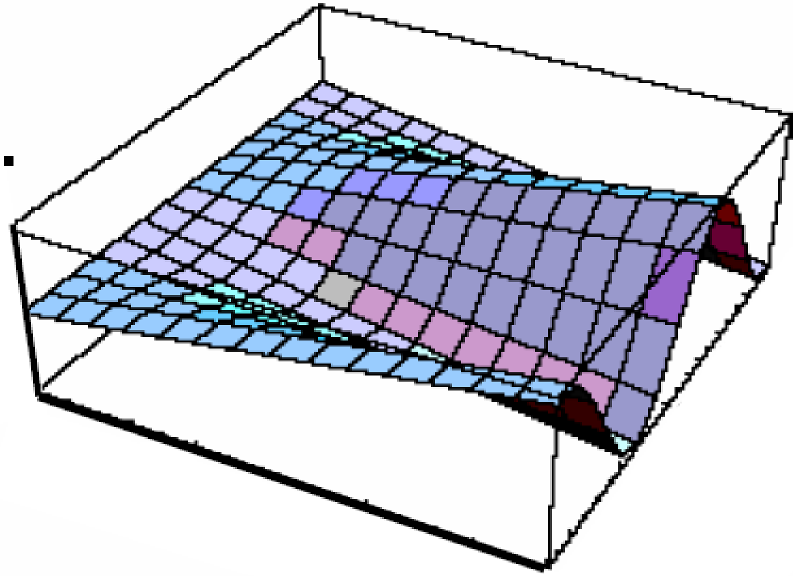
which has a DIRECTIONAL ATTRIBUTE.

The rate of change of ψ with distance s is a scalar given by

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi(\vec{r})}{\delta s} = \lim_{\delta s \to 0} \frac{\psi(\vec{r}+\delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

This 'rate' ('slope') depends on the direction in which the displacement $\delta\vec{r}$ is considered.

- **Ratio of two scalar quantities.**
- **It has a 'directional attribute'**



$$\frac{d\psi}{ds} = \lim_{\delta s\to 0}\frac{\delta\psi}{\delta s} = \lim_{\delta s\to 0}\frac{\psi(\vec{r}+\delta\vec{r})-\psi(\vec{r})}{\delta s}$$

$$\psi(\vec{\mathrm{r}})=\psi(x,y,z)$$
$$\psi(\vec{\mathrm{r}})=\psi(\rho,\varphi,z)$$
$$\psi(\vec{\mathrm{r}})=\psi(r,\theta,\varphi)$$

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x+\frac{\partial\psi}{\partial y}\delta y+\frac{\partial\psi}{\partial z}\delta z$$

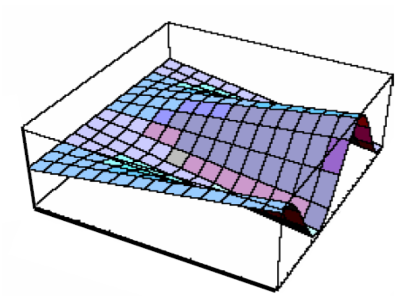
$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial z}\frac{dz}{ds}$$

$$= \frac{\partial\psi}{\partial\rho}\delta\rho+\frac{\partial\psi}{\partial\varphi}\delta\varphi+\frac{\partial\psi}{\partial z}\delta z$$

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial\rho}\frac{d\rho}{ds}+\frac{\partial\psi}{\partial\varphi}\frac{d\varphi}{ds}+\frac{\partial\psi}{\partial z}\frac{dz}{ds}$$

$$= \frac{\partial\psi}{\partial r}\delta r+\frac{\partial\psi}{\partial\theta}\delta\theta+\frac{\partial\psi}{\partial\varphi}\delta\varphi$$

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial r}\frac{dr}{ds}+\frac{\partial\psi}{\partial\theta}\frac{d\theta}{ds}+\frac{\partial\psi}{\partial\varphi}\frac{d\varphi}{ds}$$



The directional derivative

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(\vec{r}+\delta\vec{r})-\psi(\vec{r})}{\delta s}$$

Cartesian Coordinate System

$$\psi(\vec{\mathrm{r}})=\psi(x,y,z)$$

Cylindrical Polar

$$\psi(\vec{\mathrm{r}})=\psi(\rho,\varphi,z)$$

Spherical Polar

$$\psi(\vec{\mathrm{r}})=\psi(r,\theta,\varphi)$$

The directional derivative - written very nicely

- by using the expressions for the displacement $\overrightarrow{dr}$

in various coordinate systems.

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\psi(\vec{r} + \delta\vec{r}) - \psi(\vec{r})}{\delta s}$$

$$\psi(\vec{\mathrm{r}}) = \psi(x, y, z)$$

Cartesian Coordinate System

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial z}\delta z$$

$$\frac{\delta\psi}{\delta s} = \frac{\partial\psi}{\partial x}\frac{\delta x}{\delta s} + \frac{\partial\psi}{\partial y}\frac{\delta y}{\delta s} + \frac{\partial\psi}{\partial z}\frac{\delta z}{\delta s}$$

$$\delta\vec{r} = \hat{e}_x \delta x + \hat{e}_y \delta y + \hat{e}_z \delta z$$

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \lim_{\delta s \to 0} \frac{\partial\psi}{\partial x}\frac{\delta x}{\delta s} + \frac{\partial\psi}{\partial y}\frac{\delta y}{\delta s} + \frac{\partial\psi}{\partial z}\frac{\delta z}{\delta s}$$

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z = \vec{B} \bullet \vec{A}$$

$$\frac{d\psi}{ds} = \frac{\overrightarrow{dr}}{ds} \bullet \left[\hat{\mathrm{e}}_x \frac{\partial\psi}{\partial x} + \hat{\mathrm{e}}_y \frac{\partial\psi}{\partial y} + \hat{\mathrm{e}}_z \frac{\partial\psi}{\partial z} \right]$$

$$\vec{\nabla}\psi$$

$$\psi = \psi(\vec{r}) = \psi(x, y, z)$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz$$

$$d\vec{r} = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz$$

$$d\psi = \left(\hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz\right) \bullet \left[\hat{e}_x \frac{\partial \psi}{\partial x} + \hat{e}_y \frac{\partial \psi}{\partial y} + \hat{e}_z \frac{\partial \psi}{\partial z}\right]$$

$$\left[\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right] \psi = \vec{\nabla} \psi$$

$$d\psi = \overrightarrow{dr} \bullet \vec{\nabla} \psi$$

$$\overrightarrow{dr} = ?$$

$$d\left(\left|\vec{r}\right|\hat{r}\right)$$

Cartesian Unit vectors
are constant vectors;

but unit vectors of the cylindrical polar and the spherical polar coordinate systems are not!

$$\overrightarrow{dr} = ?$$

$$d\left(|\vec{r}|\,\hat{r}\right)$$

cylindrical polar coordinate

$$\frac{\partial \hat{e}_\rho}{\partial \rho} = 0, \frac{\partial \hat{e}_\rho}{\partial \varphi} = \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial \rho} = 0, \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$

spherical polar coordinate

$$\frac{\partial \hat{e}_r}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \varphi} = \sin\theta\,\hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

$$\frac{\partial \hat{e}_\theta}{\partial \varphi} = \cos\theta\,\hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_\varphi}{\partial \theta} = \vec{0}$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\theta\,\hat{e}_\theta - \sin\theta\,\hat{e}_r$$

Position and Displacement vectors in various coordinate systems

$$\vec{r} = x\hat{e}_x + y\ \hat{e}_y + z\hat{e}_z$$

$$d\vec{r} = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz$$

$$\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z$$

$$d\vec{r} = (d\rho)\hat{e}_\rho + \rho(d\hat{e}_\rho) + (dz)\hat{e}_z$$

$$d\vec{r} = \hat{e}_\rho d\rho + \hat{e}_\varphi \rho d\varphi + \hat{e}_z dz$$

$$\vec{r} = r\hat{e}_r$$

$$d\vec{r} = (dr)\hat{e}_r + r(d\hat{e}_r)$$

$$d\vec{r} = (dr)\hat{e}_r + r\left[\frac{\partial\hat{e}_r}{\partial\theta}d\theta + \frac{\partial\hat{e}_r}{\partial\varphi}d\varphi\right]$$

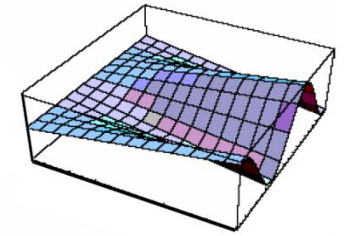
$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\varphi\hat{e}_\varphi$$

In order to avoid making careless mistakes, always try to write unit vectors first, differential elements last!

Example:

$$d\vec{r} = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\varphi r\sin\theta d\varphi$$

Gradient in the Cartesian Coordinate System



$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\psi(\vec{r}+\delta\vec{r})-\psi(\vec{r})}{\delta s}$$

$$d\psi = \overrightarrow{dr} \bullet \vec{\nabla}\psi$$

$$\frac{d\psi}{ds} = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial z}\frac{dz}{ds}$$

$$d\vec{r} = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz$$

$$\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\vec{\nabla}\psi = \left[\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right]\psi$$

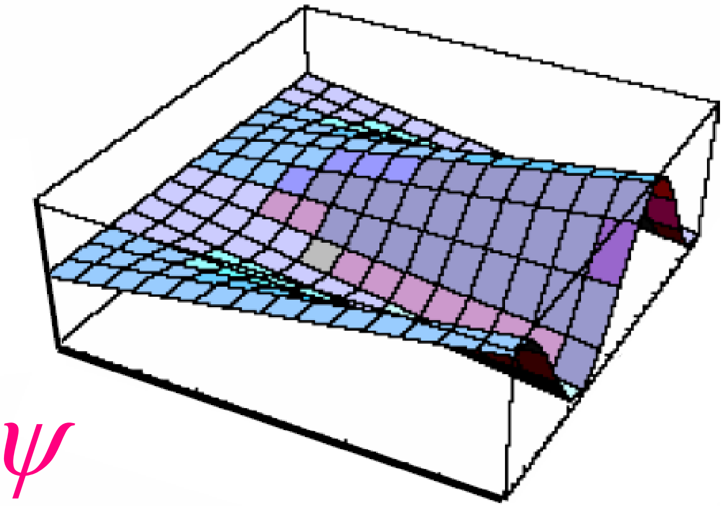
$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

$\delta s = \left|\overrightarrow{\delta r}\right|$, tiny increament

$ds = \left|\overrightarrow{dr}\right|$, differential increament

The GRADIENT $\vec{\nabla}$ of a scalar point function $\psi(\vec{r})$ yields a vector point function such that the component of the resultant vector along any direction (given by a unit vector $\hat{u}$) gives the DIRECTIONAL DERIVATIVE $\frac{d\psi}{ds}$ of the scalar function in the direction of that unit vector.



$\frac{d\psi}{ds}$ Directional Derivative

$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

The GRADIENT $\vec{\nabla}$ of a scalar point function $\psi(\vec{r})$ yields a vector point function such that the component of the resultant vector along any direction (given by a unit vector $\hat{u}$) gives the DIRECTIONAL DERIVATIVE $\frac{d\psi}{ds}$ of the scalar function in the direction of that unit vector.

This definition is **independent** of the coordinate system used.

$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}; \qquad \delta s = \left|\overrightarrow{\delta r}\right|; \qquad ds = \left|\overrightarrow{dr}\right|$$

$$\frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta \psi}{\delta s} = \hat{u} \bullet \vec{\nabla} \psi = \frac{\overrightarrow{dr}}{ds} \bullet \vec{\nabla} \psi$$

$$\psi(\vec{r} + \overrightarrow{\delta r}) - \psi(\vec{r}) = \delta\psi = \overrightarrow{\delta r} \bullet \vec{\nabla} \psi$$

This definition of (a) the directional derivative and (b) the gradient is **independent** of the coordinate system used.

Cylindrical Polar Coordinate System

$$\delta\psi = \delta\psi(\rho,\varphi,z)$$
$$= \delta\rho\frac{\partial\psi}{\partial\rho} + \delta\varphi\frac{\partial\psi}{\partial\varphi} + \delta z\frac{\partial\psi}{\partial z}$$
$$= \overrightarrow{\delta r} \bullet \vec{\nabla}\psi$$
$$\left(\hat{e}_\rho\delta\rho + \hat{e}_\varphi\rho\delta\varphi + \hat{e}_z\delta z\right) \bullet \vec{\nabla}\psi$$

How should we express $\vec{\nabla}\psi$ *such that:*

$$\delta\psi = \overrightarrow{\delta r} \bullet \vec{\nabla}\psi$$

where:

$$\delta\vec{r} = \hat{e}_\rho\delta\rho + \hat{e}_\varphi\rho\delta\varphi + \hat{e}_z\delta z$$

Following form of the gradient operator will work!

$$\vec{\nabla} = \hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{e}_z\frac{\partial}{\partial z}$$

Note how the ρ *cancels* $\frac{1}{\rho}$

$$\delta\psi =$$
$$\left(\hat{e}_\rho\delta\rho + \hat{e}_\varphi\rho\delta\varphi + \hat{e}_z\delta z\right) \bullet \left(\hat{e}_\rho\frac{\partial}{\partial\rho} + \hat{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{e}_z\frac{\partial}{\partial z}\right)\psi$$

Spherical Polar Coordinate System

$$\delta\psi = \delta\psi(r,\theta,\varphi)$$
$$= \delta r\frac{\partial\psi}{\partial r} + \delta\theta\frac{\partial\psi}{\partial\theta} + \delta\varphi\frac{\partial\psi}{\partial\varphi}$$
$$= \overrightarrow{\delta r} \bullet \vec{\nabla}\psi$$
$$\left[\hat{e}_r(\delta r) + \hat{e}_\theta(r\delta\theta) + \hat{e}_\varphi(r\sin\theta\ \delta\varphi)\right] \bullet \vec{\nabla}\psi$$

How should we <u>express</u> $\vec{\nabla}\psi$

such that:

$$\delta\psi = \overrightarrow{\delta r} \bullet \vec{\nabla}\psi$$

where:

$$\delta\vec{r} = \hat{e}_r(\delta r) + \hat{e}_\theta(r\delta\theta) + \hat{e}_\varphi(r\sin\theta\ \delta\varphi)$$

Following form of the gradient operator will work!

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

Note the cancellation of the factors that are circled

$$\delta\psi =$$
$$\left[\hat{e}_r(\delta r) + \hat{e}_\theta(r\delta\theta) + \hat{e}_\varphi(r\sin\theta\ \delta\varphi)\right] \bullet \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\psi$$

Consolidated expressions for the GRADIENT

Cartesian Coordinate System

$$\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$$

Cylindrical Polar Coordinate System

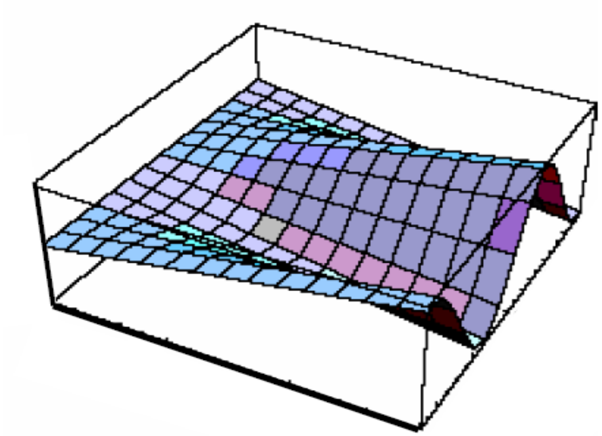
$$\vec{\nabla} = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\frac{d\psi}{ds} = \hat{u} \bullet \vec{\nabla}\psi$$

$$\hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

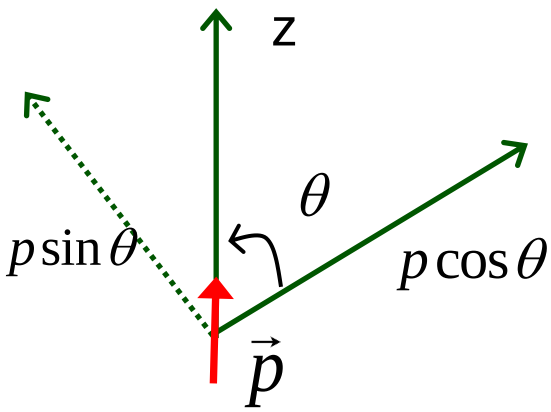
$$\delta s = \left|\overrightarrow{\delta r}\right|, \text{ tiny increament}$$

$$ds = \left|\overrightarrow{dr}\right|, \text{ differential increament}$$



Spherical Polar Coordinate System

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$



The elctrostatic potential due to a point dipole is

$$U(r,\theta,\varphi)=\frac{k\ \vec{r}\cdot\vec{p}}{r^2}=\frac{kpr\cos\theta}{r^2}\ \text{where } k=\frac{1}{4\pi\varepsilon_0}$$

$$\vec{\nabla}U=\left[\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]U$$

$$\vec{E}=-\vec{\nabla}U$$

$$E_r=-\frac{\partial U}{\partial r}=\frac{2kp\cos\theta}{r^3};\qquad E_\theta=-\frac{1}{r}\frac{\partial U}{\partial\theta}=\frac{kp\sin\theta}{r^3}$$

$$E_\varphi=-\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}=0\Rightarrow\quad \vec{E}(r,\theta,\varphi)=\frac{kp}{r^3}(\hat{e}_r\ 2\cos\theta+\hat{e}_\theta\sin\theta)$$

since $\vec{p}=\hat{e}_r\ p\cos\theta-\hat{e}_\theta\ p\sin\theta,$

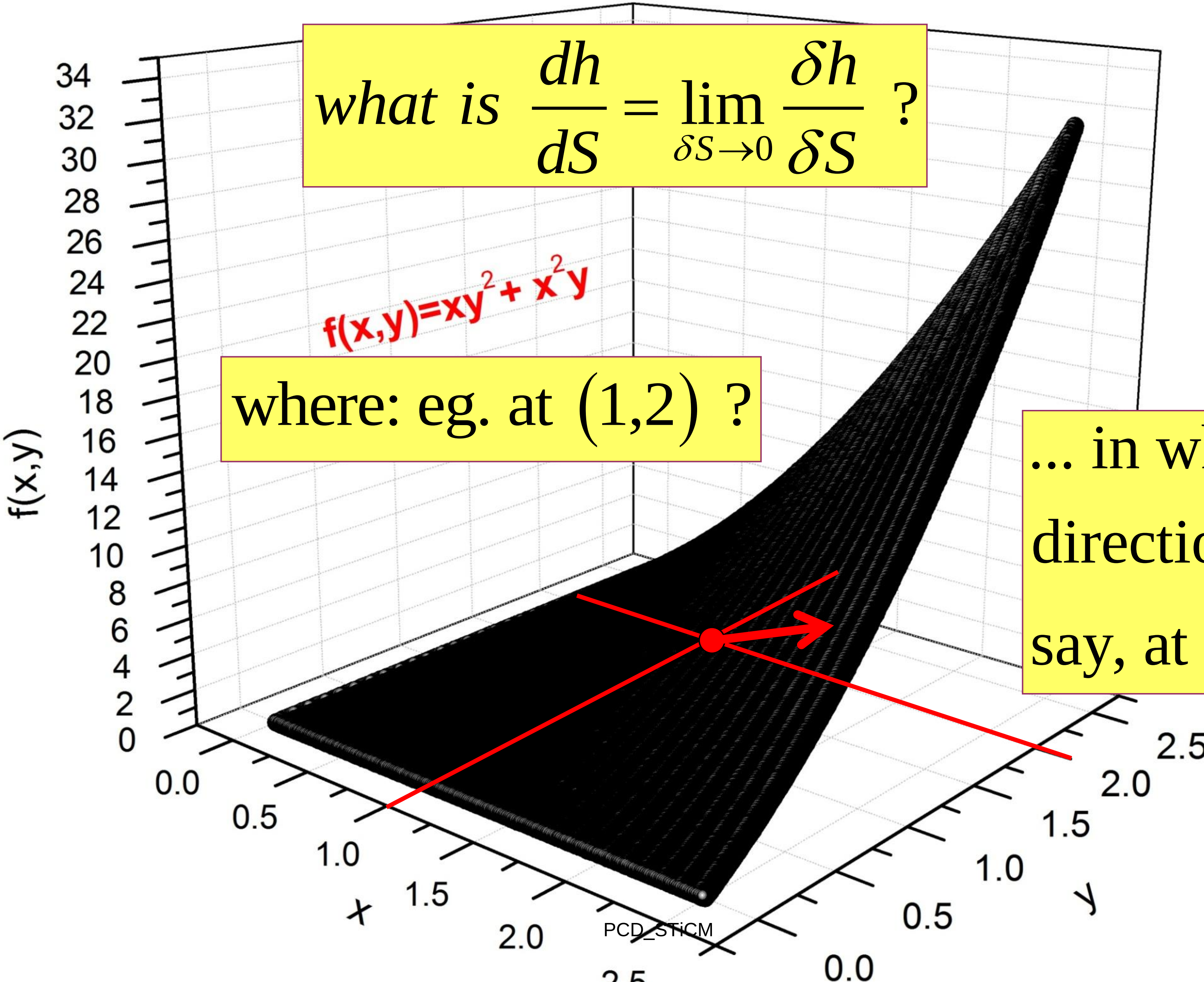
$$\vec{E}=k\frac{3(\vec{p}\cdot\hat{e}_r)\hat{e}_r-\vec{p}}{r^3}$$

what is $\dfrac{dh}{dS} = \lim_{\delta S \to 0} \dfrac{\delta h}{\delta S}$?

where: eg. at $(1,2)$?

... in which direction ?

say, at 45^0 ?



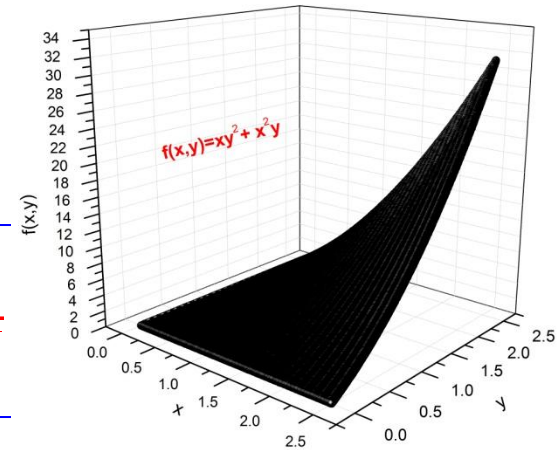
$$f(x,y) = xy^2 + x^2 y \qquad \vec{\nabla} f(x,y) = \vec{\nabla}\left(xy^2 + x^2 y\right)$$

$$\vec{\nabla} f(x,y) = \hat{e}_x \frac{\partial}{\partial x}\left(xy^2 + x^2 y\right) + \hat{e}_y \frac{\partial}{\partial y}\left(xy^2 + x^2 y\right)$$

$$\vec{\nabla} f(x,y) = \hat{e}_x \left(y^2 + 2xy\right) + \hat{e}_y \left(2xy + x^2\right)$$

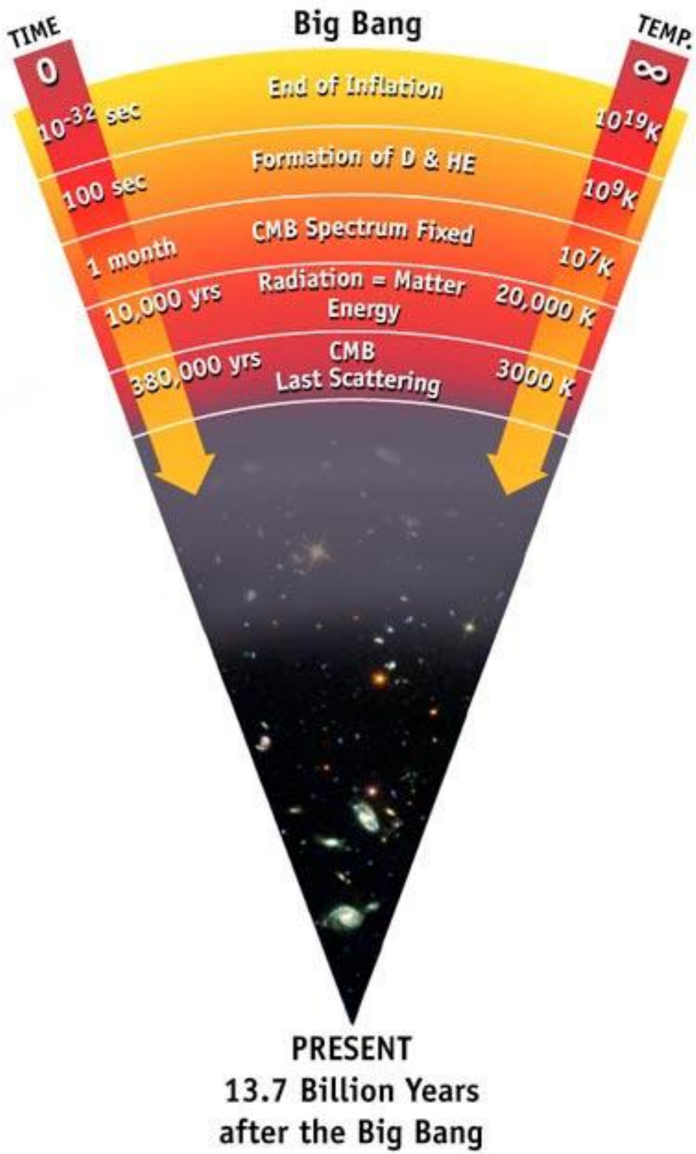
$$\frac{df}{ds} = \hat{u} \bullet \vec{\nabla} f; \qquad \hat{u} = \lim_{\delta s \to 0} \frac{\overrightarrow{\delta r}}{\delta s} = \frac{\overrightarrow{dr}}{ds}$$

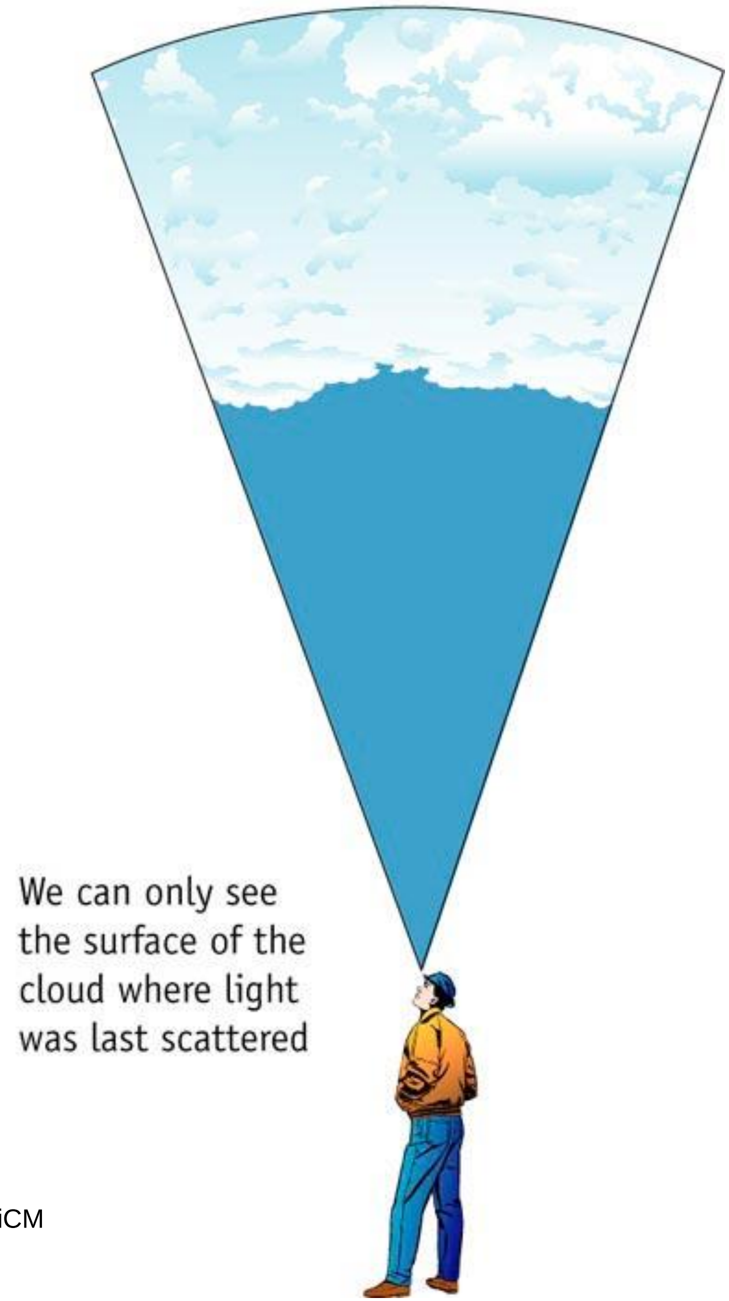
$$\hat{u} = \hat{e}_x \cos\frac{\pi}{4} + \hat{e}_y \sin\frac{\pi}{4} = \hat{e}_x \frac{1}{\sqrt{2}} + \hat{e}_y \frac{1}{\sqrt{2}}$$



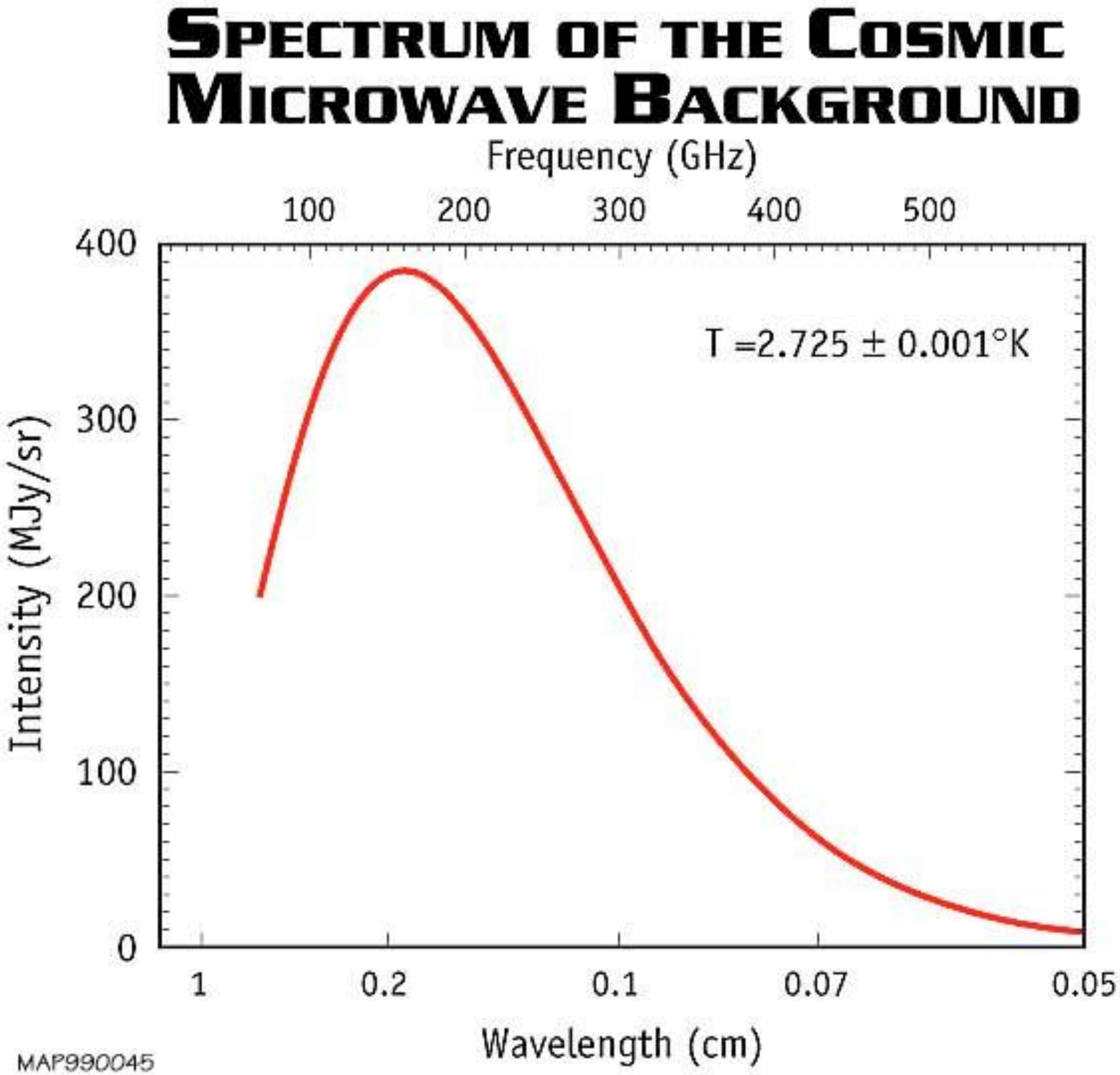
$$\left[\frac{df}{ds}\right]_{(1,2)} = \left[\frac{1}{\sqrt{2}}\left(y^2 + 2xy\right) + \frac{1}{\sqrt{2}}\left(2xy + x^2\right)\right]_{(1,2)}$$

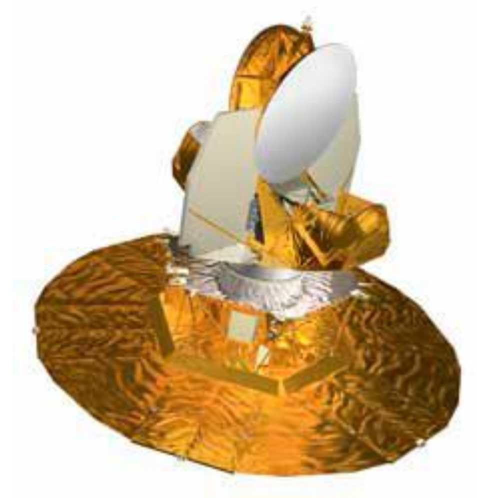
http://map.gsfc.nasa.gov/media/990053/990053sb.jpg 23/June/2010





The cosmic microwave background Radiation's "surface of last scatter" is analogous to the light coming through the clouds to our eye on a cloudy day.





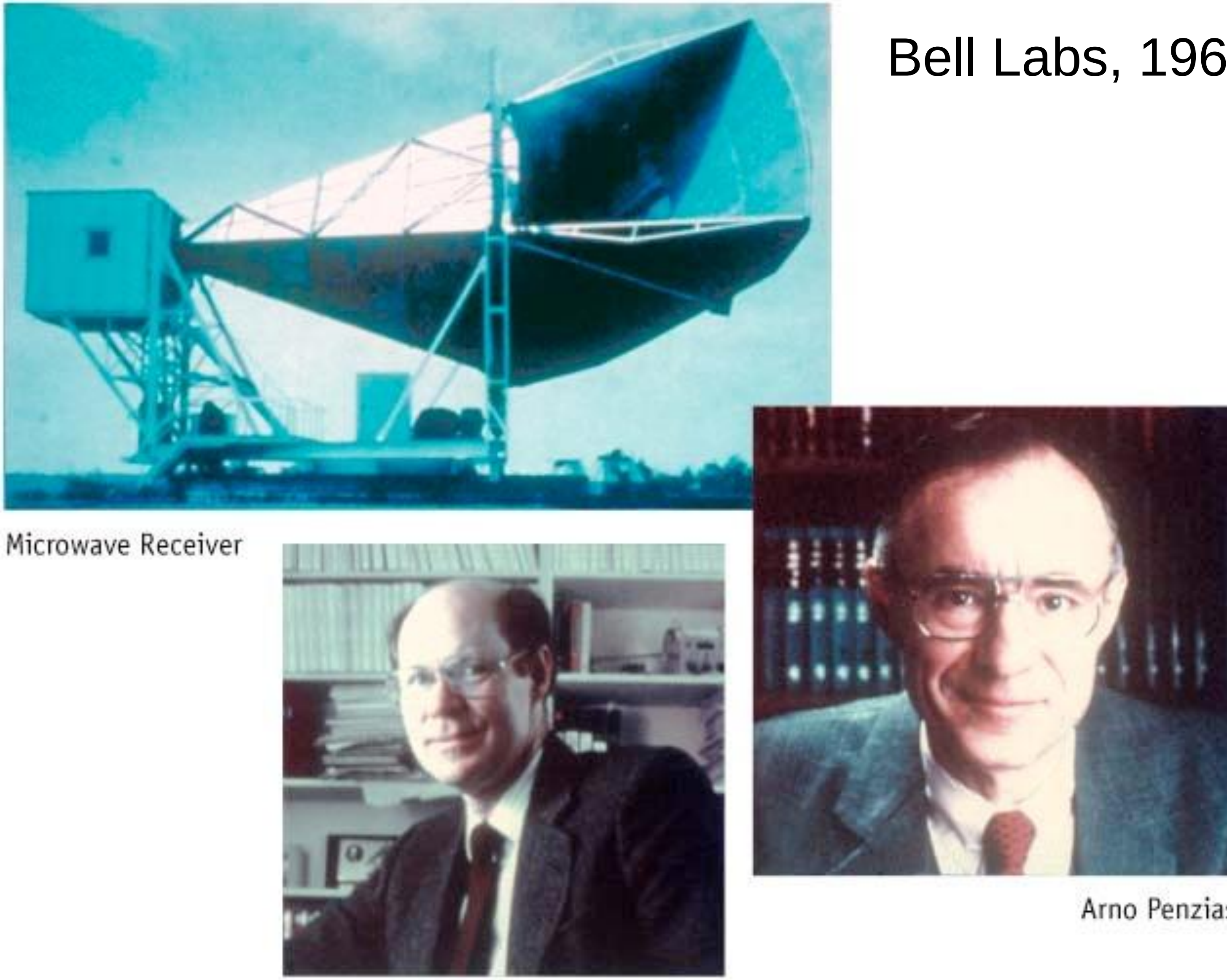
Wilkinson
Microwave
Anisotropy Probe

launched on
June 30, 2001

http://map.gsfc.nasa.gov/media/ContentMedia/990015b.jpg

Discovery of Cosmic Background

Bell Labs, 1965



http://map.gsfc.nasa.gov/media/ContentMedia/990045b.jpg

Potentials, Gradients, Fields………..

Applications….

gravitational fields……..

electromagnetic fields…….

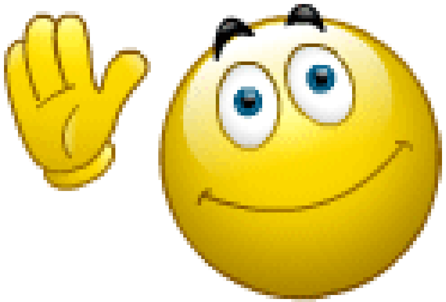
Microwave Cosmic Background Radiation Asymmetry…

Questions ? Comments ?

We shall take a break here…

pcd@physics.iitm.ac.in

http://www.physics.iitm.ac.in/~labs/amp/



STiCM

Select / Special Topics in Classical Mechanics

P. C. Deshmukh

Department of Physics
Indian Institute of Technology Madras
Chennai 600036

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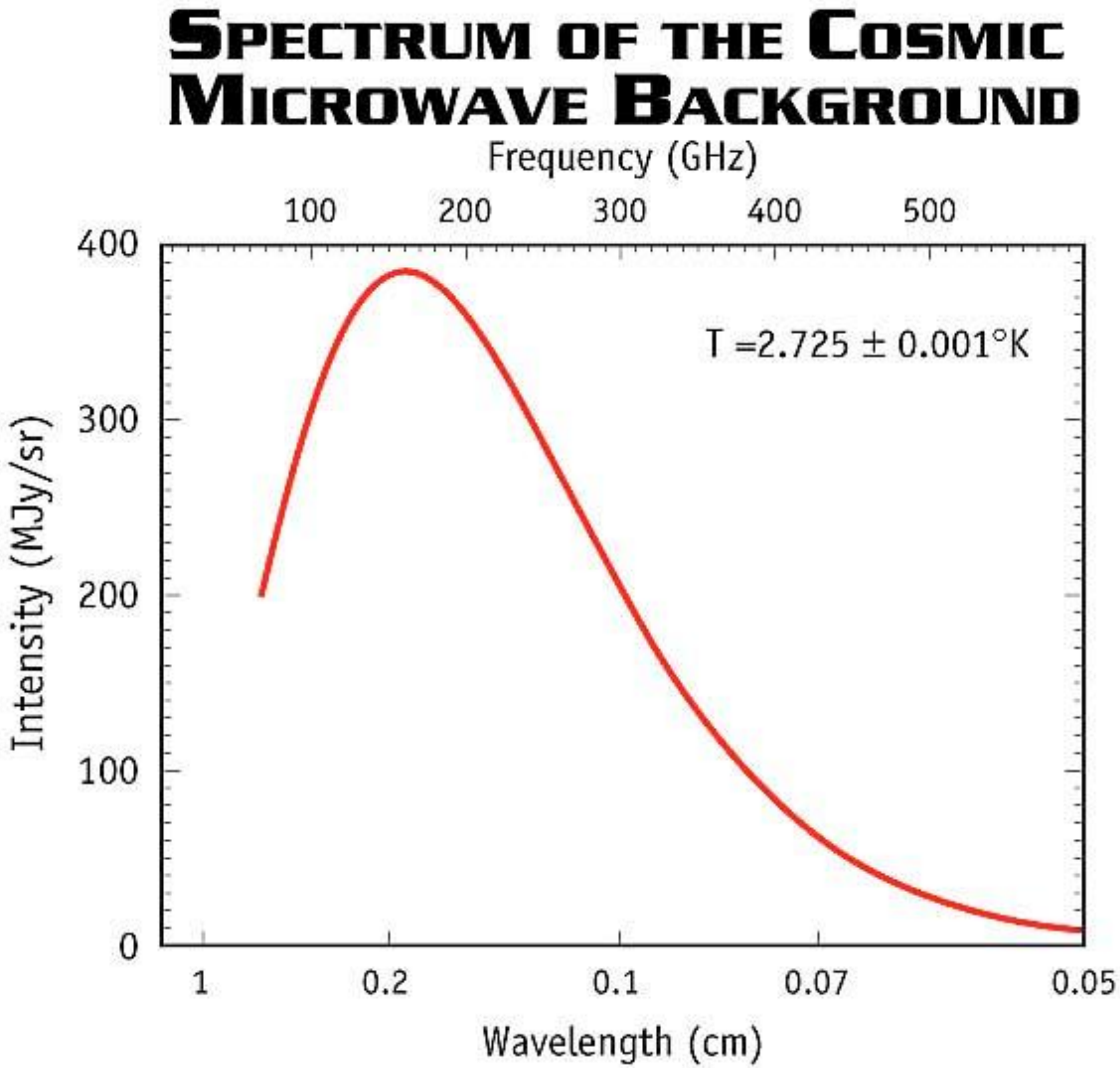
STiCM Lecture 25

Unit 7 : Potentials, Gradients, Fields



Bell Labs, 1965

Arno Penzias

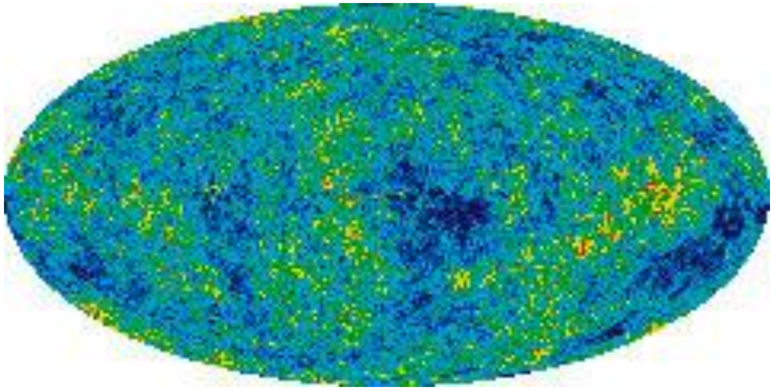
Robert Wilson

Jy: 10^{-26} watt per square meter per hertz http://map.gsfc.nasa.gov/news/index.html

MEASURING COSMIC ASYMMETRY?

The average temperature is 2.725 Kelvin (degrees above absolute zero; equivalent to -270 C or -455 F).

The cosmic microwave temperature fluctuations from the 5-year WMAP satellite data seen over the full sky.



Red regions are warmer and blue regions are colder by about ***0.0002 degrees***.

Geostationary orbits

Communication

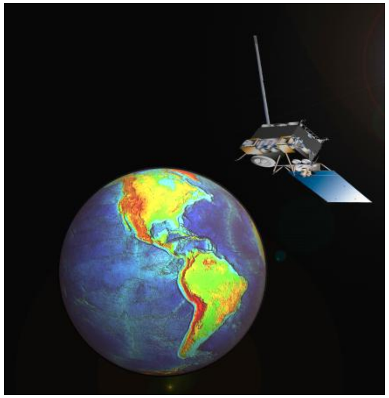
Radio/TV broadcasting

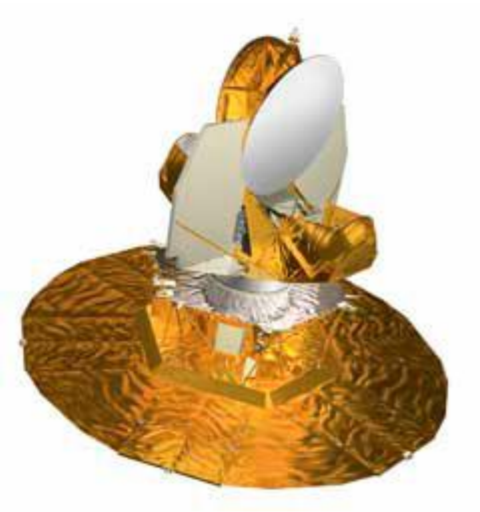
Meteorology,

Weather forecasting….

GPS

Remote sensing…





WMAP

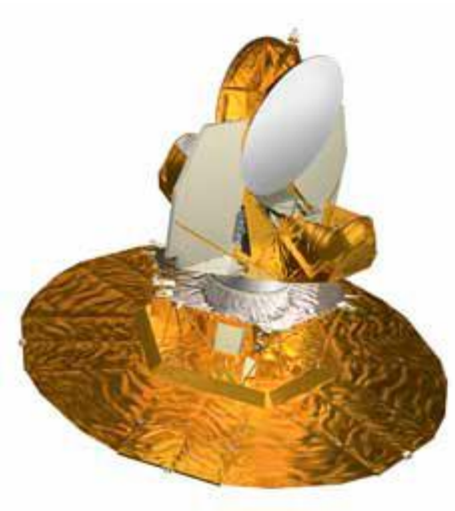
Wilkinson Microwave Anisotropy Probe

launched on June 30, 2001

WMAP

launched on
June 30, 2001

Wilkinson
Microwave
Anisotropy Probe



Sun, Earth,

& Moon

are always behind the instrument's

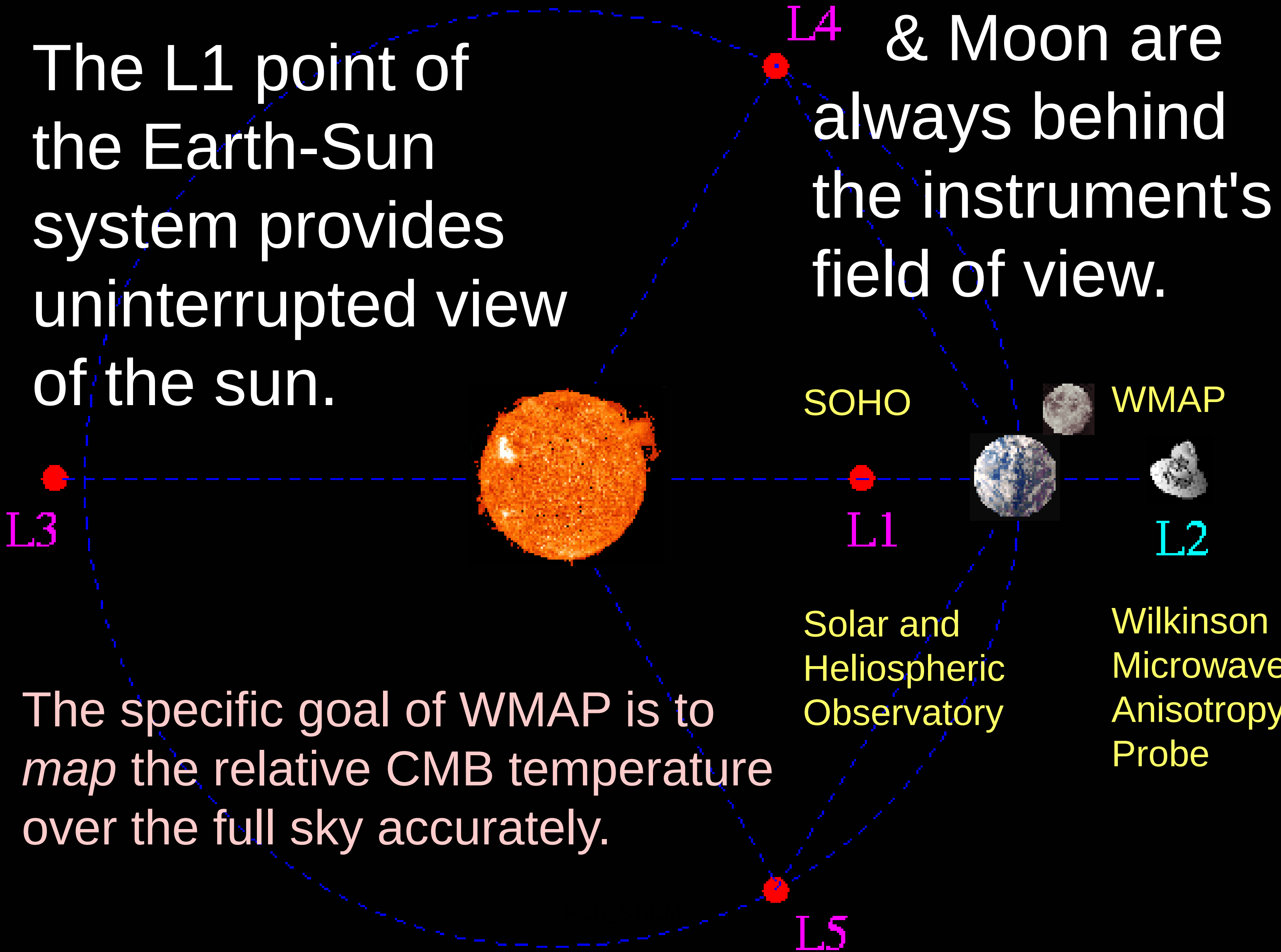
field of view.

at ‘L2’

http://map.gsfc.nasa.gov/mission/observatory.html

The L1 point of the Earth-Sun system provides uninterrupted view of the sun.

At L2: Sun, Earth, & Moon are always behind the instrument's field of view.



The specific goal of WMAP is to *map* the relative CMB temperature over the full sky accurately.

WMAP: Wilkinson Microwave Anisotropy Probe

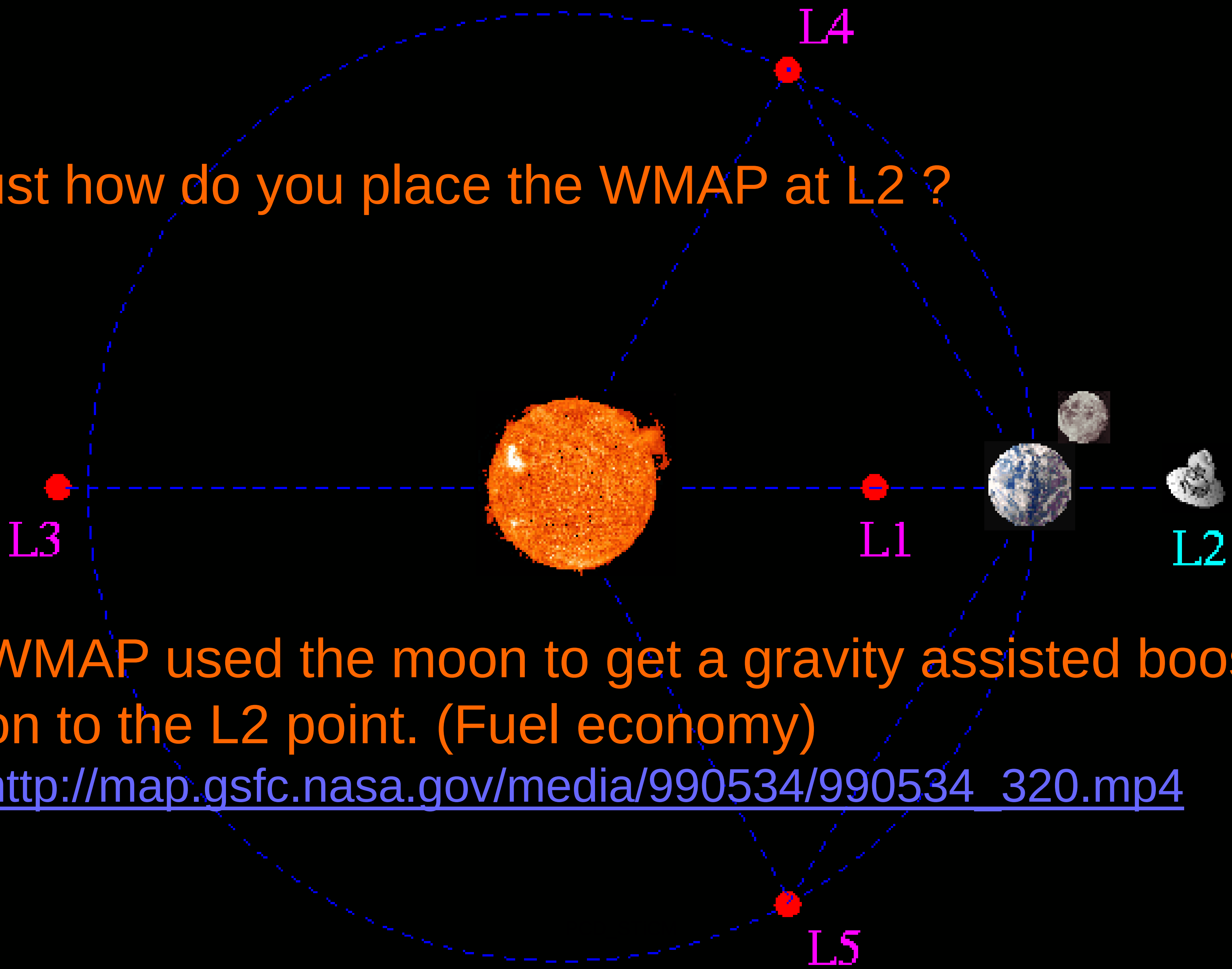
Ref.: http://map.gsfc.nasa.gov/mission/observatory.html / October 19, 2009 / 10:48pm /

Orbits around the Lagrange point L2 of the Sun-Earth system.

At the Lagrange points, the resultant gravitational pull of the sun and the earth on an object, such as a satellite, is precisely equal to the centripetal force that causes that object to rotate with them.

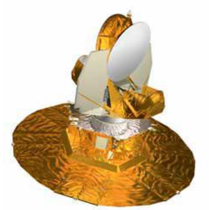
Lagrange (~1772) found five such points ('**LAGRANGE POINTS**') for the sun-earth system using the ***PRINCIPLE OF EXTREMUM ACTION***.

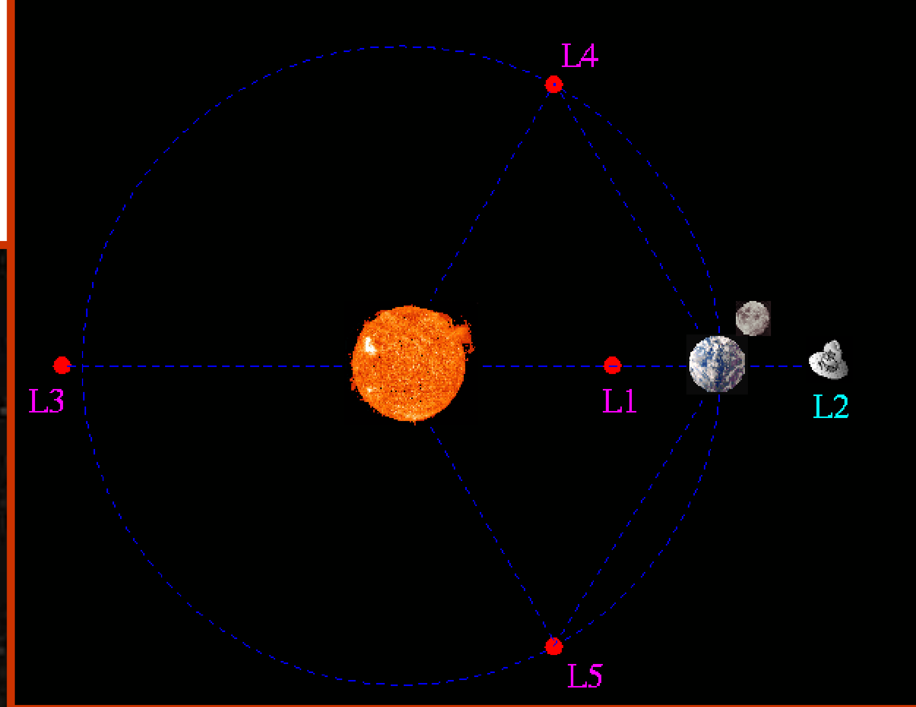
Just how do you place the WMAP at L2 ?

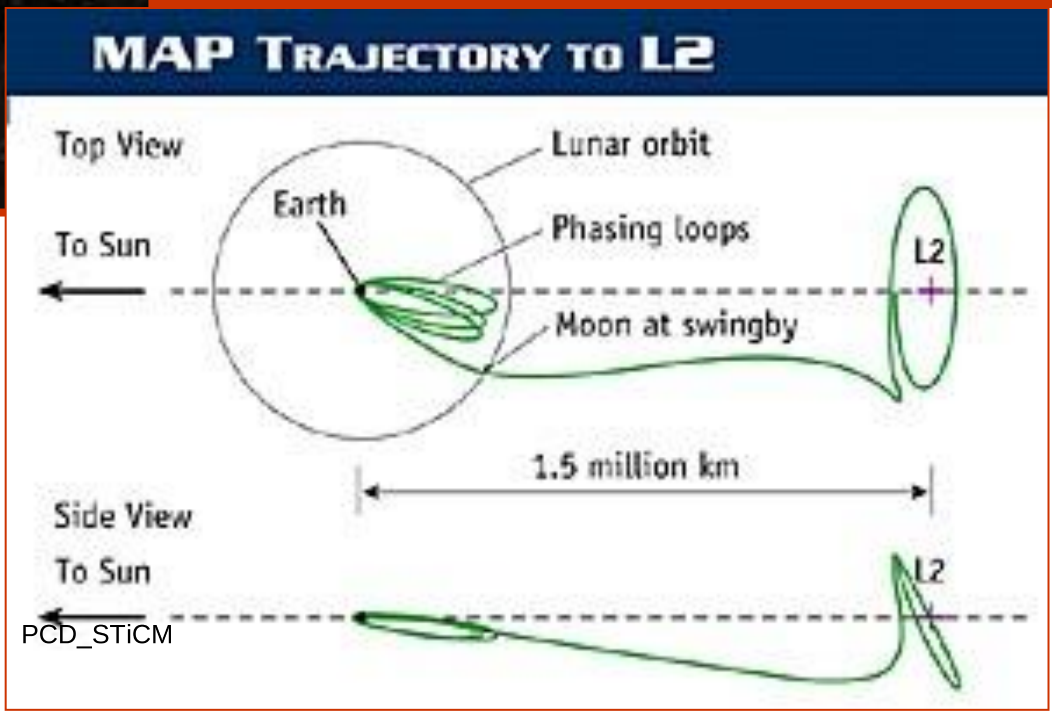


WMAP used the moon to get a gravity assisted boost on to the L2 point. (Fuel economy)

http://map.gsfc.nasa.gov/media/990534/990534_320.mp4







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Wilkinson Microwave Anisotropy Probe: ***WMAP***

WMAP_990534_320_MOON_SWINGS_WMAP.wmv

http://map.gsfc.nasa.gov/media/990534/index.html

23 June 2010

WMAP_990533_320_WMAP_ORBITING_L2.wmv

http://map.gsfc.nasa.gov/media/990533/index.html

June 23, 2010

L1, L2, L3: Unstable ; L4, L5: Stable

Derivation of the L1, L2, L3 points

Reference:

http://www-istp.gsfc.nasa.gov/stargaze/Slagrang.htm

Derivation of the L4 and L5 points, based on "When Trojans and Greeks Collide" by I. Vorobyov, "Quantum," p. 16-19, Sept-Oct. 1999.

Uses rotating frames of reference.

Reference:

http://www-istp.gsfc.nasa.gov/stargaze/Slagrng3.htm

Interesting Reading on ‘Lagrangian Points’

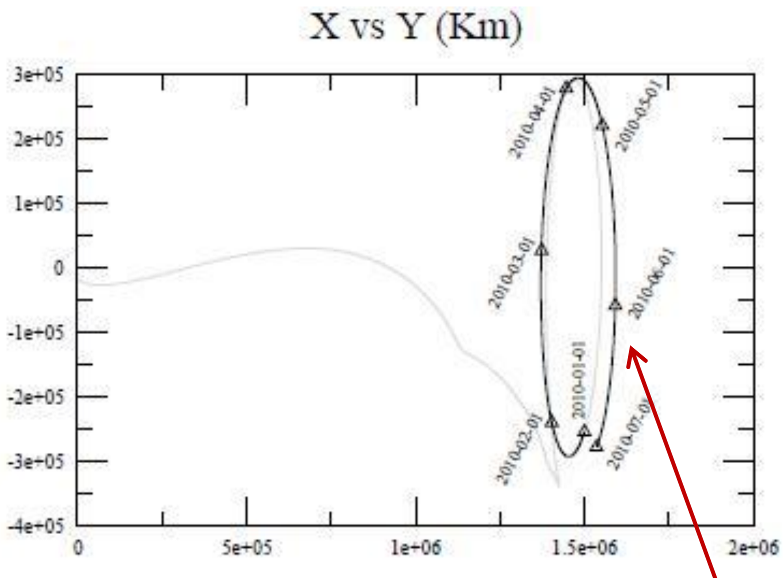
The colonization of space

Reference:

Gerard K. O'Neill

Physics Today, **27(9)**:32-40 (September, 1974)

http://www.aeiveos.com/~bradbury/Authors/Engineering/ONeill-GK/TCoS.html



June 1, 2010

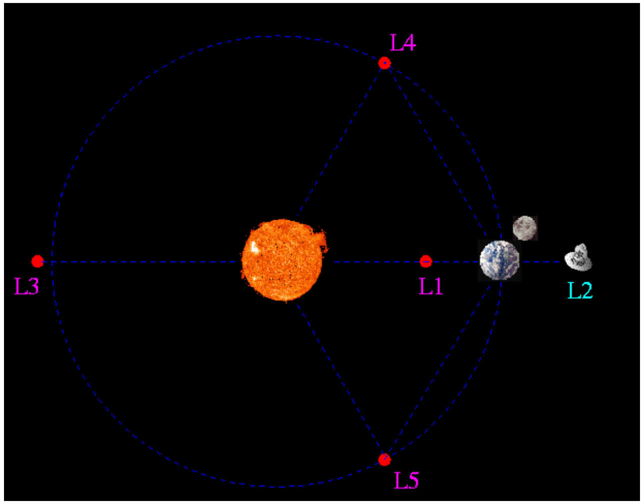
Planck surveyor: Europe's first mission to study the relic radiation from the Big Bang, the cosmic microwave background radiation (CMB).

http://esamultimedia.esa.int/docs/planck/Planck-Factsheet.pdf

Solar & Heliospheric Observatory: SOHO

An Uninterrupted View of the Sun

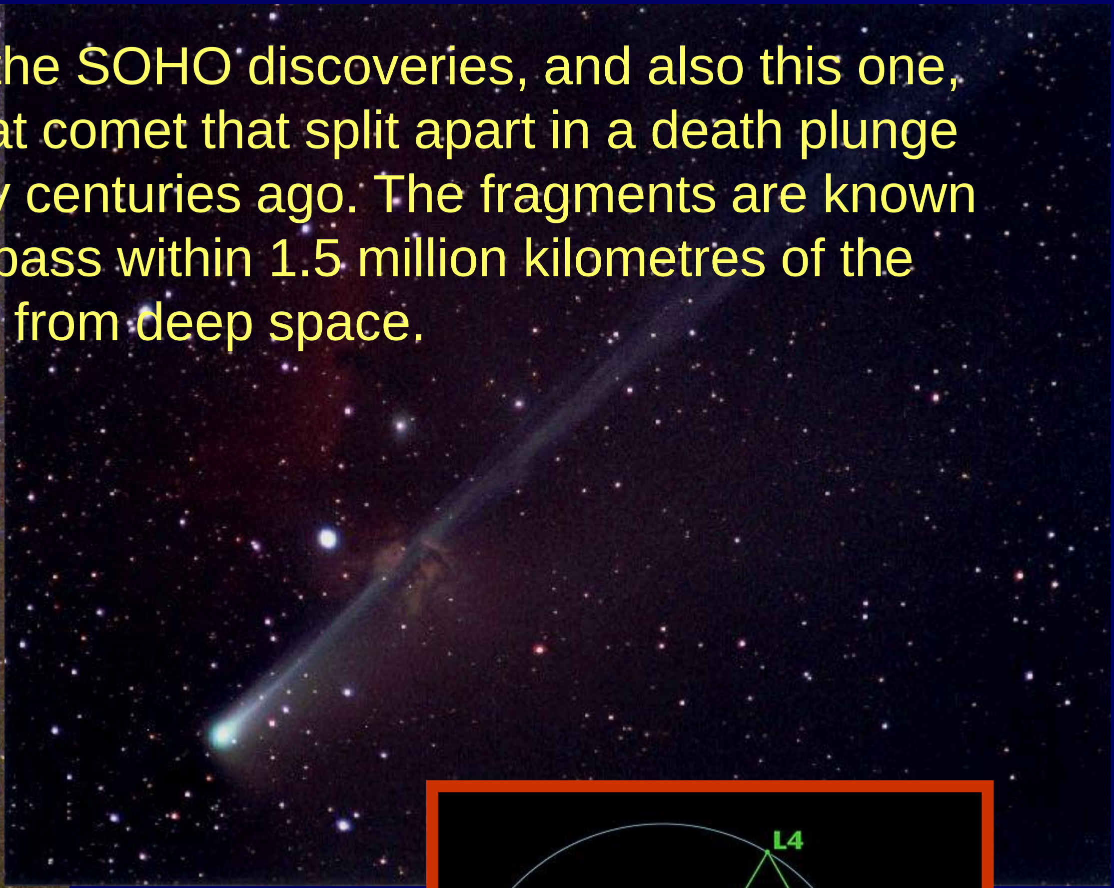
SOHO moves around the Sun in step with the Earth, by slowly orbiting around the First Lagrangian Point (L1).

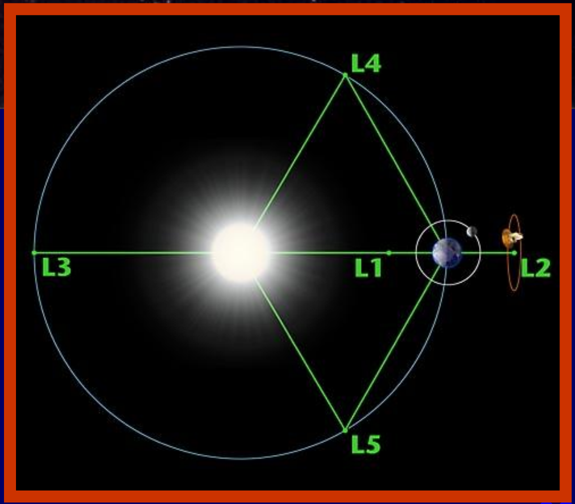


- where the combined gravity of the Earth and Sun keep SOHO in an orbit locked to the Earth-Sun line.

http://www.nasa.gov/mission_pages/soho/index.html http://www.aerith.net/comet/catalog/1998J1/pictures.html

Roughly eighty-five percent of the SOHO discoveries, and also this one, are fragments from a once great comet that split apart in a death plunge around the Sun, probably many centuries ago. The fragments are known as the **Kreutz group** and now pass within 1.5 million kilometres of the Sun's surface when they return from deep space.

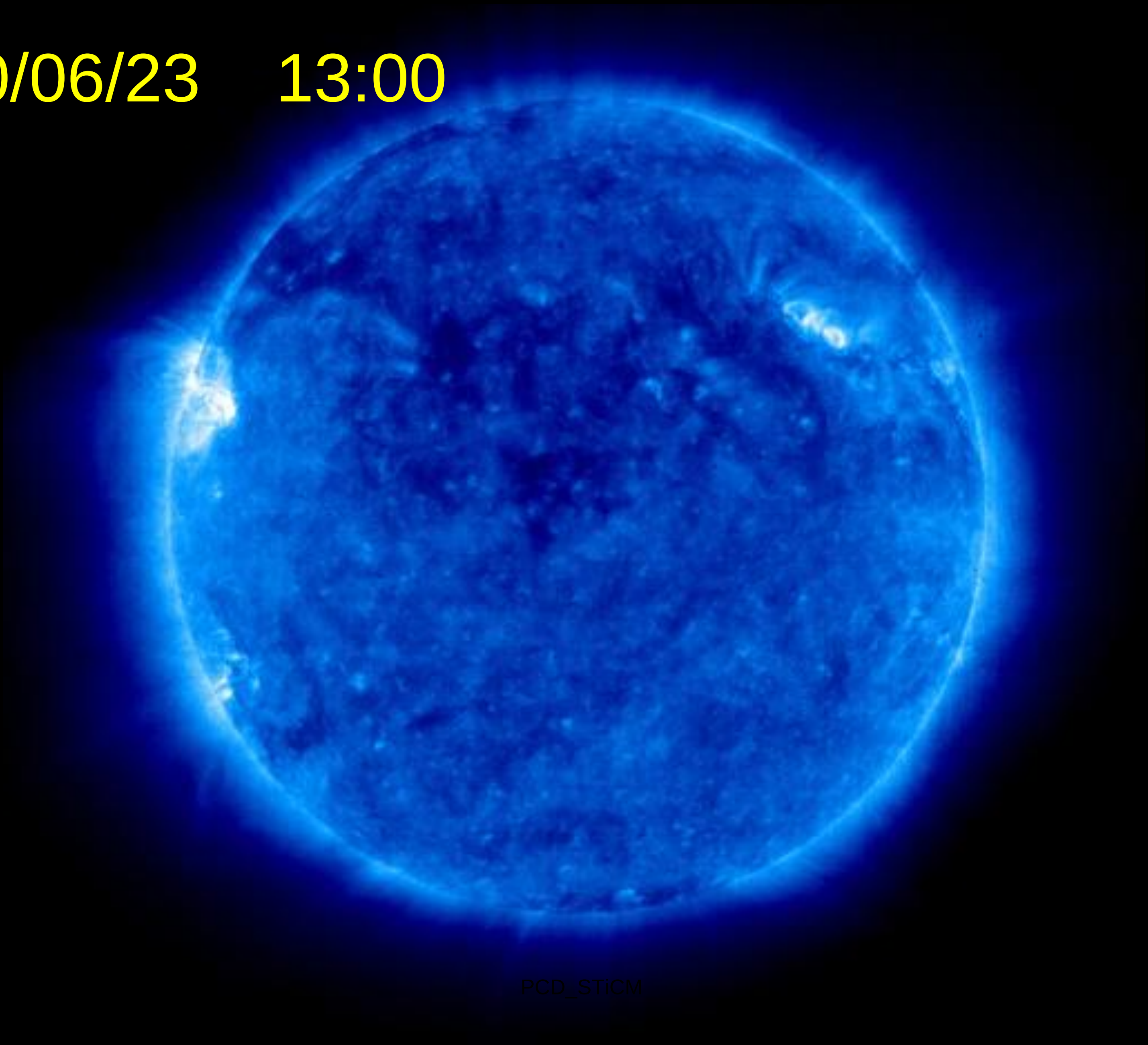




Downloaded on
20th Oct. 2009
08:00am
Pictures by SOHO@L1

http://sohowww.nascom.nasa.gov/data/realtime/eit_171/512/

2010/06/23 13:00



Solar & Heliospheric Observatory: SOHO

An Uninterrupted View of the Sun

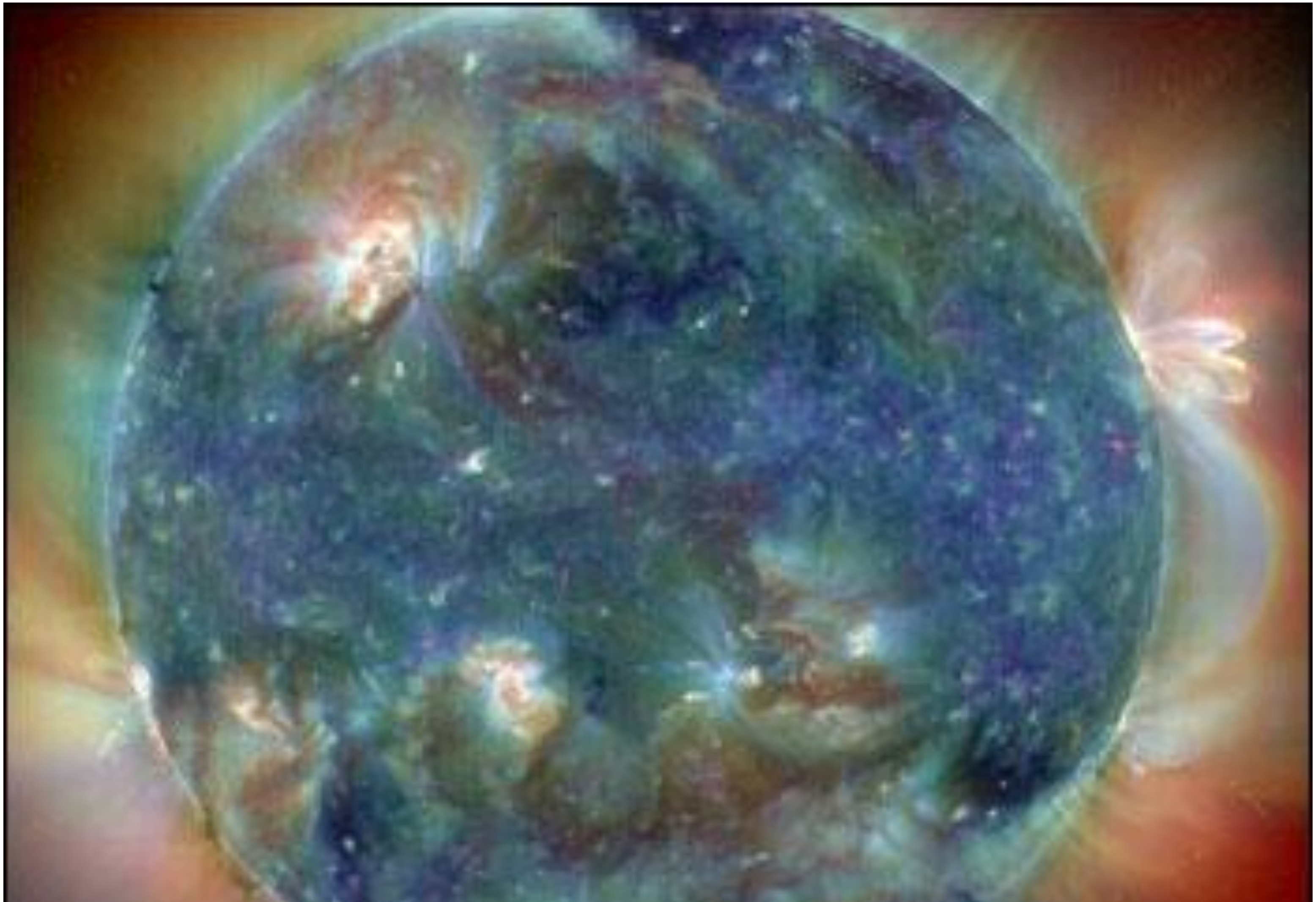
The L1 point is approximately 1.5 million kilometers away from Earth (about four times the distance of the Moon), in the direction of the Sun.

SOHO was launched on December 2, 1995


SOHO_WMV V9.wmv

http://www.nasa.gov/mission_pages/soho/index.html

"Absence of sunspots make scientists wonder if they're seeing a calm before a storm of energy"



By Stuart Clark New Scientist Tuesday, June 22, 2010

Pictures from SOHO@L1

http://www.washingtonpost.com/wp-dyn/content/article/2010/06/21/AR2010062104114.html?g=0

June 24, 2010

PCD_STiCM

Gradient Potentials Fields

$$\vec{F} = -\vec{\nabla} U$$

………… an example….

Given: Force experienced by a particle is

$$\vec{F}(\rho,\varphi,z) = -\hat{e}_\rho\ \rho\cos 2\varphi + \hat{e}_\varphi\ \rho\sin 2\varphi + \hat{e}_z\ z$$

Obtain the potential for the given field.

$$\vec{F} = -\vec{\nabla}U = -\hat{e}_\rho \frac{\partial U}{\partial \rho} - \hat{e}_\varphi \frac{1}{\rho}\frac{\partial U}{\partial \varphi} - \hat{e}_z \frac{\partial U}{\partial z}$$

$$i.e.\ \frac{\partial U}{\partial \rho} = \rho\cos 2\varphi \qquad \Rightarrow\ \mathrm{U} = \frac{\rho^2}{2}\cos 2\varphi + f(\varphi, z)$$

$$\Rightarrow\ \frac{\partial U}{\partial \varphi} = -\rho^2 \sin 2\varphi + \frac{\partial f}{\partial \varphi}.$$

Given: $\vec{F}(\rho,\varphi,z) = -\hat{e}_\rho\ \rho\cos 2\varphi + \hat{e}_\varphi\ \rho\sin 2\varphi + \hat{e}_z\ z$

Obtain the potential for the given field.

$$\vec{F} = -\vec{\nabla}U = -\hat{e}_\rho \frac{\partial U}{\partial \rho} - \hat{e}_\varphi \frac{1}{\rho}\frac{\partial U}{\partial \varphi} - \hat{e}_z \frac{\partial U}{\partial z}$$

$$U = \frac{\rho^2}{2}\cos 2\varphi + f(\varphi, z)\ ; \Rightarrow \quad \frac{\partial U}{\partial \varphi} = -\rho^2 \sin 2\varphi + \frac{\partial f}{\partial \varphi}.$$

$$\hat{e}_\varphi \bullet \vec{F} = -\frac{1}{\rho}\frac{\partial U}{\partial \varphi} = \rho \sin 2\varphi; \quad \frac{\partial U}{\partial \varphi} = -\rho^2 \sin 2\varphi$$

$$\therefore \quad \frac{\partial f}{\partial \varphi} = 0 \ \Rightarrow \ f = f(z) + c$$

$$\frac{\partial f}{\partial z} = \frac{\partial U}{\partial z} = -z$$

$$\Rightarrow \quad f = \frac{-z^2}{2} + c$$

$$U(\rho,\varphi,z) = \frac{\rho^2}{2}\cos 2\varphi - \frac{z^2}{2} + c \ ?$$

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

Eugene P.. Wigner: Communications in Pure and Applied Mathematics, Vol. 13, No. I (February 1960).
THE UNREASONABLE EFFECTIVENSS OF MATHEMATICS IN THE NATURAL SCIENCES

Connection between 'potential' and 'field'

Choice of 'gauge'

Mere calculus ?

Different potentials often give the same field; 'same physics'

Gauge transformation vs. Symmetry transformation

We shall take a break here…….

Questions ? Comments ?

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Our aim:

- introduce you to the romance in physics,
- beauty in its simplicity,
- and rigor in its formulation.

Next L26 : Unit 8
Gauss' Law, Equation of Continuity
Hydrodynamic / Electrodynamic illustrations.